

# Mathematical Tables *and other* Aids to Computation

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Edited by

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# Mathematical Tables

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## Aids to Computation



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## Coupon Collector's Test for Random Digits

**1. Introduction.** Increasing use of random numbers, especially in Monte Carlo procedures and in large computing installations, has served to focus attention on the various tests for randomness. KENDALL and BABINGTON-SMITH<sup>1</sup> list four tests for so-called local randomness. While not giving the coupon collector's test (to be described below) a place in their now classical list of four tests, they did use a modified coupon collector's test in some of their investigations.

In an ordered set of digits, say, one may count the length of a sequence, beginning at a specified position, necessary to give or include the complete set of all ten digits. Or one may count the length required to give a set of  $k$ ,  $k < 10$ , different digits. The distribution of these observed lengths for different initial positions can then be compared with a theoretically computed distribution. Such a test will be called the coupon collector's test from an analogy with certain sales promotion schemes.

The theoretical distribution may be computed from formulas given by H. von SCHELLING,<sup>2</sup> which formulas hold for the case where the individual category probabilities might be unequal. For the random digital case with category probabilities equal to  $1/10$ , von SCHELLING's formulas simplify readily and may be conveniently related to the "differences of zero." These latter quantities are tabulated by FISHER and YATES<sup>3</sup> up to sequences of length 26. But in using the coupon collector's test for a complete set of all 10 digits it has been found that the mean of the length distribution is slightly greater than 29, and a table of probabilities associated with the sequence lengths 10 to 26 inclusive would hardly give a realistic picture of the entire distribution.

The present author has therefore extended this tabulation, and exact probabilities are given for sequence lengths  $10 \leq n \leq 35$  and approximate probabilities for sequence lengths  $36 \leq n \leq 75$ . The probabilities were computed from the relation

$$p_n = \frac{1}{10^{n-1}} \sum_{j=0}^q (-1)^j \binom{q}{j} (q-j)^{n-1},$$

and are listed in Table 3.

If one were interested in sequence lengths necessary to obtain five different digits, the mean of this distribution is approximately 6.46. The range 5 to 26 inclusive available from FISHER and YATES<sup>3</sup> might be sufficient here.

**2. Sequence Lengths for Decimal Expansions of  $\pi$  and  $e$ .** It would be a simple matter to program a large digital computing machine so that it would tabulate the distribution of the sequence lengths needed for complete sets for a given ordered digital collection. However, the author did not have such a digital computing machine available, and he made a tabulation by hand for the decimal expansion of  $\pi$ . The 2035 decimal approximation to  $\pi$  given by GEORGE W. REITWIESNER<sup>4</sup> was used as the raw material for this count. Beginning with the initial position 3 in  $\pi \approx 3.14159 \dots$ , it was recorded that a sequence length of 33

positions was needed to get all the ten digits. Beginning anew with the thirty-fourth position digit (which is a 2), it was recorded that a sequence of 18 positions was needed to get a complete set of all the 10 digits. Continuing this procedure, 67 sequences of complete sets were obtained, plus an incomplete sequence (at the end of the decimal expansion) of length 15. It was considered advisable to make the sequences non-overlapping as described above since there is considerable dependence among the set of sequence lengths if every position in the decimal expansion of  $\pi$  is regarded as a new starting point.

The sequence lengths for  $\pi$  are also included in Table 3. This tabulation for  $\pi$  was checked by Mr. WAYNE JONES of the Department of Defense, Washington, D. C.

Mr. JONES also made a tabulation based on the decimal expansion of  $e$ . REITWIESNER<sup>4</sup> gave a 2010 decimal approximation to  $e$ . An additional 490 places was given by METROPOLIS, REITWIESNER and von NEUMANN.<sup>5</sup> Mr. JONES found 82 complete sequences using 2486 digits in the expansion of  $e$ . This tabulation is also given in Table 3. The author desires to thank Mr. JONES for this count.

**3. Statistical Tests.** The mean and the standard deviation of the theoretical distribution may be computed from results given by von SCHELLING<sup>3</sup> or FELLER.<sup>6</sup> These theoretical values and the corresponding observed values for  $\pi$  and  $e$  are given below.

TABLE 1

	Theoretical	Observed	
		$\pi$	$e$
Mean	29.29	30.16	30.32
Standard deviation	11.21	11.83	10.64

To use a chi-square test, it is desirable that the expected values all exceed 10 in size. Since the sample size for  $\pi$  is small (67) some grouping of the sequence lengths is necessary to meet this desired minimum. The following results were obtained for a convenient grouping.

TABLE 2

Sequence lengths, $n$	$\pi$		$e$	
	Observed	Expected	Observed	Expected
10-19	13	11.604	12	14.202
20-23	13	11.720	11	14.344
24-27	9	11.491	14	14.064
28-32	5	11.480	15	14.050
33-39	13	10.195	17	12.477
40 and over	14	10.510	13	12.863
Totals	67	67.000	82	82.000
Chi-squared test values	6.436		2.826	

Neither of these chi-square test values is unusually out of line. It has been previously reported<sup>6,7</sup> that (using a sample of 2000 digits for  $e$ ) excessive flatness in the single frequencies was noted, and an indication was obtained that the single digits in  $e$  are "non-random."<sup>8</sup> Apparently, this phenomenon did not reflect itself

TABLE 3

*Table of Probabilities and Empirical Distributions*

<i>n</i>	<i>p<sub>n</sub></i>	Observed for	
		<i>r</i>	<i>e</i>
10	.0003 6288	0	0
11	.0016 3296	0	0
12	.0041 9126 4	0	0
13	.0080 9315 2	0	0
14	.0130 4560 8576	0	1
15	.0186 3435 9744	1	2
16	.0243 5958 6451 2	1	1
17	.0297 8461 8864	6	2
18	.0345 7819 0373 1264	3	2
19	.0385 2892 7611 5744	2	4
20	.0415 3577 5577 4998 4	2	3
21	.0435 8654 2461 1780 8	2	2
22	.0447 3311 6259 6932 2752	3	3
23	.0450 6836 4358 6388 8896	6	3
24	.0447 0706 5704 2485 9072	2	3
25	.0437 7151 9771 4451 888	1	1
26	.0423 8153 3617 2618 0440 544	3	5
27	.0406 4806 4094 7299 5986 8	3	5
28	.0386 6968 0608 0430 0677 8256	2	3
29	.0365 3106 7596 0890 3842 8456	2	1
30	.0343 0291 1584 0099 4076 1298 5728	0	4
31	.0320 4266 1164 2497 4751 5573 3056	1	2
32	.0297 9578 2029 8315 1051 7926 5414 4	0	5
33	.0275 9724 1577 4565 5030 9198 2083 2	1	2
34	.0254 7304 5949 3494 9321 3424 0393 4016	2	2
35	.0234 4171 8456 6112 5667 0619 1553 5264	2	3
36	.0215 1565 5696 6141 0012	3	3
37	.0197 0233 0293 7275 2140	2	5
38	.0180 0533 0690 9430 5978	3	1
39	.0164 2524 1844 5333 7918	0	1
40	.0149 6037 8429 7183 4300	1	3
41	.0136 0738 6073 5944 1433	1	1
42	.0123 6172 7525 9630 8189	1	1
43	.0112 1807 0507 6953 1223	6	1
44	.0101 7059 2895 9431 7444	1	0
45	.0092 1321 9356 5092 1003	1	1
46	.0083 3980 1802 1014 6739	0	0
47	.0075 4425 4318 8464 5255	0	1
48	.0068 2065 1566 0407 8968	0	1
49	.0061 6329 8170 0629 7386	0	0
50	.0055 6677 5325 1020 3197	0	0

TABLE 3—Continued

n	$p_n$	Observed for	
		$\pi$	$\epsilon$
51	.0050 2596 9683 5475 6580	0	1
52	.0045 3608 8658 5862 1077	0	0
53	.0040 9266 5455 6666 7056	0	1
54	.0036 9155 6480 2793 7180	0	0
55	.0033 2893 3218 7175 0148	0	0
56	.0030 0127 0238 7102 7949	0	0
57	.0027 0533 0592 0946 3793	0	0
58	.0024 3814 9607 8519 0648	0	0
59	.0021 9701 7828 5704 8789	1	0
60	.0019 7946 3656 1777 4941	1	0
61	.0017 8323 6124 7283 4517	0	0
62	.0016 0628 8101 7164 6560	0	0
63	.0014 4676 0128 6430 1251	0	0
64	.0013 0296 5041 3524 9640	0	0
65	.0011 7337 3456 8177 1422	0	0
66	.0010 5660 0172 2241 9129	0	1
67	.0009 5139 1491 6632 0338	0	0
68	.0008 5661 3473 2828 0290	0	0
69	.0007 7124 1073 6049 1625	0	0
70	.0006 9434 8154 4810 4916	0	0
71	.0006 2509 8310 7043 5014	0	0
72	.0005 6273 6471 7289 2795	0	1
73	.0005 0658 1228 5628 1531	0	0
74	.0004 5601 7836 1436 4579	0	0
75	.0004 1049 1841 9142 4169	0	0
76	.0036 9745 8744 5702 0432	1	(77) 0
and over		Total 67	82

in materially changing the characteristics of the sequence length distribution for the coupon collector's test. Some question arises as to whether the single frequency test and the coupon collector's test are independent, and also which test has the greater power.

The chi-square test values in Table 2 were calculated by assuming that the sequence lengths for complete sets of digits are independent draws from a known (infinite) multinomial probability distribution. (Null hypothesis.) The alternatives would include unspecified sorts of dependency and other underlying probabilities different from those given in Table 3.

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<sup>1</sup> M. G. KENDALL & B. BABINGTON SMITH, "Randomness and random sampling numbers," *Royal Stat. Soc., Jn.*, v. 101, 1938, p. 147-166.

<sup>2</sup> H. von SCHELLING, "Auf der Spur des Zufalls," *Deutsches Statistisches Zentralblatt*, v. 26, 1934, p. 137-146. Also, "Coupon collecting for unequal probabilities," *Amer. Math. Mon.*, v. 61, 1954, p. 306-311.

<sup>3</sup> R. A. FISHER & F. YATES, *Statistical Tables for Biological Agriculture and Medical Research*, 3rd edition, London, 1948, Table XXII.

<sup>4</sup> GEORGE W. REITWIESNER, "An ENIAC determination of  $\pi$  and  $e$  to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11-15.

<sup>5</sup> N. C. METROPOLIS, G. REITWIESNER, & J. von NEUMANN, "Statistical treatment of values of first 2000 decimal digits of  $e$  and  $\pi$  calculated on the ENIAC," *MTAC*, v. 4, 1950, p. 109-111.

<sup>6</sup> W. FELLER, *Probability Theory and Its Applications*. Volume 1, New York, 1950, p. 175-181.

<sup>7</sup> F. GRUENBERGER, "Further statistics on the digits of  $e$ ," *MTAC*, v. 6, 1952, p. 123-134.

## A Method for the Evaluation of a System of Boolean Algebraic Equations

With the advent of large scale electronic devices whose logical design is described by a system of Boolean algebraic equations, a method to mechanize the evaluation of such a system and shorten this evaluation with respect to time will be increasingly useful. Such a method will be described in this paper.

The problem may be described as follows: Given a set of  $n$  variables,  $Q^k$ , ( $k = 1, 2, \dots, n$ ) each of which may take on the value 1 (true) or 0 (false) at any time  $t$ ; then the value of any  $Q^k$  at time  $t + 1$  may be defined by the system of Boolean equations

$$(1) \quad R_t^k = f_k(Q_t^e)$$

$$(2) \quad S_t^k = g_k(Q_t^e)$$

$$(3) \quad Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k)$$

where  $1 \leq q \leq n$ . For example, the recirculation loop of a dynamic flip-flop may be defined simply by

$$Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k) = R_t^k.$$

In another system, a more complex definition

$$Q_{t+1}^k = \phi(Q_t^k, R_t^k, S_t^k) = Q_t^k \cdot \bar{R}_t^k \cdot \bar{S}_t^k + R_t^k \cdot \bar{S}_t^k + \bar{Q}_t^k \cdot R_t^k \cdot S_t^k$$

may be taken, where  $R_t^k$  and  $S_t^k$  are the two inputs to flip-flop  $Q^k$ .

We shall use the symbols for conjunction, disjunction, and negation

$Q^1 \cdot Q^2$	" $Q^1$ and $Q^2$ "	conjunction
$Q^1 + Q^2$	" $Q^1$ or $Q^2$ "	disjunction
$\bar{Q}^1$	"Not $Q^1$ "	negation

which are defined by the truth tables<sup>1</sup>

$Q^1$	$Q^2$	$Q^1 \cdot Q^2$	$Q^1 + Q^2$	$\bar{Q}^1$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

A "term" is defined as one or more variables conjoined together, e.g.,  $Q^1 \cdot Q^2 \cdot \bar{Q}^3$ ; and an "equation" as  $M$  terms,  $T_m$ , ( $m = 1, 2, \dots, M$ ) disjoined together, e.g.,  $Q^1 \cdot Q^2 \cdot \bar{Q}^3 + Q^1 \cdot Q^4$ . Now note that the value of a term is zero if any variable in

that term has the value zero, and the value of an equation is one if any term in that equation has the value one. We will deal only with the case where all of the expressions on the right are in normal disjunctive form, but this represents no restriction on the method since any equation in the above form may be so written.

Our method for solving this system of equations involves the use of punched cards. While this description is in terms of IBM punched cards, the method is, of course, not restricted to this type of card. Cards and a key punch are the only tools required. In fact, the key punch is required only to initially punch the cards as described below, and is not used thereafter in the actual process of evaluation.

Two types of cards must be defined and punched in order to implement the evaluation. Cards of type I each represent the  $m^{\text{th}}$  term of the equation for  $R^k$  or  $S^k$  and are labeled " $R^kT_m$ " or " $S^kT_m$ "; these are the "equation" cards. Cards of type II represent the value of  $Q^k$  (1 or 0) and are labeled  $Q^kT$  or  $Q^kF$ , respectively; these are the "value" cards and one or the other is selected for use at time  $t$  for each  $k$ . The problem may now be restated as: given the set of cards of type II containing the appropriate card  $Q^kT$  or  $Q^kF$  for all  $Q^k$  at time  $t$ , and further given the equations as represented by the deck of cards of type I, construct a set of cards of type II representing the value of each  $Q^k$  at time  $t + 1$ .

The twelve rows and 80 columns of the IBM card may be thought of as a single pair of rows and 480 "columns." Let this column number correspond to  $k$  and let us represent variable  $Q^k$  by one of three possible combinations of punches or no punches in the pair of rows of column  $k$ .

Thus cards of type I are constructed such that the card  $R^kT_m$  has column  $q$  punched as follows:

- i) a punch in row 1 if  $Q^q$  is a variable in term  $m$  of the equation for  $R^k$ ,
- ii) a punch in row 2 if  $\bar{Q}^q$  is a variable in term  $m$  of the equation for  $R^k$ , or,
- iii) no punch in column  $q$  if neither  $Q^q$  nor  $\bar{Q}^q$  appears as a variable in term  $m$  of the equation for  $R^k$ .

The card  $S^kT_m$  is constructed in an entirely similar way.

Cards of type II are constructed in a manner which is somewhat analogous to the above. The same 480 columns of a single pair of rows is considered. Now, however, there is one and only one punch in column  $k$ , and all rows of all other columns are punched; i.e.,

- i) the punch in column  $k$  is in row 1 if  $Q^k$  is true . . . in this case the card is labeled  $Q^kT$ ;
- ii) the punch in column  $k$  is in row 2 if  $Q^k$  is false ( $\bar{Q}^k$  is true) . . . in this case the card is labeled  $Q^kF$ ;
- iii) both rows of all other columns are punched.

A "deck" of cards of type II consists of all of the cards  $Q^kT$  and  $Q^kF$ . A "set" of these cards at time  $t$  consists of one and only one card  $Q^kT$  or  $Q^kF$  representing the value of each  $Q^k$  at time  $t$ . Thus there are twice as many cards in a deck of type II as in a set of type II. It will be convenient to have two decks of type II so that one set of cards may be made up for time  $t$  from one deck and, using this set and the entire deck of cards of type I, a set of cards of type II for time  $t + 1$  may be compiled from the other deck.

We shall call the set of cards of type II, for time  $t$ , the "this time table" and those for time  $t + 1$  the "next time table." Note that after we have formed the complete next time table, we may redefine it as the new this time table and proceed to form a new next time table for time  $t + 2$  from the (now) unused deck of type II. This process may continue as long as required, and is essentially that of evaluation of the equations.

The process is carried out as follows: The this time table is stacked and remains undisturbed (but not unused) until the next time table is completed. The entire this time table is physically aligned so that each position on the card representing each row and column is directly in line with the corresponding position of all other cards in the set. With these cards so arranged, it is easy to determine which positions are punched in every card of the set; this is most simply done visually by simply sighting through the entire deck. Further, we may now observe that we can sight through one and only one of the two rows in every column. That this is the this time table for time  $t$  has the significance that if we can sight thru row 1 of column  $k$ , then  $Q_t^k = 1$ , or, if we can sight thru row 2 of column  $k$ , then  $Q_t^k = 0$ . Thus by simply aligning the cards of the this time table, we can read the values of  $Q_t^k$  required to evaluate  $\phi$  (Eq. 3).

We may now evaluate  $R_t^k$  (Eq. 1) by the following measures if the cards  $R^k T_m$  of type I are arranged in ascending order of  $k$  and  $m$ : Place card  $R^1 T_1$  in alignment with the this time table as above. If we can sight thru the deck of cards thus formed wherever  $R^1 T_1$  is punched, then  $R_t^1 = 1$  and it is not necessary to examine cards  $R^1 T_2, R^1 T_3, \dots, R^1 T_M$ , at this time. If not, we remove  $R^1 T_1$  and proceed to examine  $R^1 T_2$  in the same way. This process continues until either 1) we find  $R^1 T_m$  satisfying the condition that we can sight thru the deck consisting of it and the this time table, in which case  $R_t^1 = 1$ , or 2) we have examined all  $R^1 T_m$  without satisfying the condition, in which case  $R_t^1 = 0$ . Having thus evaluated  $R_t^1$ , we proceed in the same manner to evaluate  $S_t^1$ . When this is done, we are now in a position to solve for  $Q_{t+1}^1$  from equation 3. In exactly the same way we proceed to evaluate in order  $R_t^2, S_t^2, Q_{t+1}^2, R_t^3, S_t^3, Q_{t+1}^3, \dots, R_t^n, S_t^n, Q_{t+1}^n$ . The values of the  $Q$ 's are recorded as the next time table by forming the proper set from the available deck of cards of type II. The values of  $R$  and  $S$  need not be retained.

The above description in which IBM cards are employed requires that  $N \leq 480$ . It is easy to see that this number may be increased by increasing the number of cards of type I and type II appropriately. However, the more usual problem is to reduce the number of cards employed when  $n$  is sufficiently less than 480, this being the more usual case.

The number of cards of type II employed is perfectly straightforward: If  $n \leq 480$ , then the number in a deck is  $2n$  and that in a set is  $n$ ; if  $480 < n \leq 960$ , then the number in a deck is  $4n$  and that in a set is  $2n$ ; and so forth. But if  $n$  is sufficiently small, e.g.,  $n = 240$ , then we may make certain types of combinations of the equation cards (type I). For example, we may represent  $R^k T_m$  and  $S^k T_m$  on the same card, or we may represent two terms of  $R^k$  on one card. Such combinations of cards serve to reduce the time required to carry out the evaluation, where  $n$  is sufficiently small, without loss of generalization. They require certain

obvious modifications of the configuration of the value cards (type II) in the form of duplication of information.

When the problem actually is solved as described above, it is convenient but not necessary to have such devices as a box of some transparent material in which to stack the this time table, perhaps with a strategically placed light source to assist in observations. In fact, the design has been completed for a device to completely mechanize the process described so that the human operator has only to feed cards to a card reader with the solution carried out and the results recorded automatically.

J. A. POSTLEY

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<sup>1</sup> HANS REICHENBACH, *Elements of Symbolic Logic*. The Macmillan Company, New York, 1947. The notation used in this paper is that of Reichenbach except that his symbol " $\sigma$ " is replaced by the symbol "+".

<sup>2</sup> E. C. NELSON, "An algebraic theory for use in digital computer design," *Transactions of the IRE Professional Group on Electronic Computers*, vol. EC-3, p. 12-21, September 1954.

<sup>3</sup> ENGINEERING RESEARCH ASSOCIATES, *High-Speed Computing Devices*. McGraw-Hill Book Company, New York, 1950.

<sup>4</sup> ARTHUR W. BURKS, DON W. WARREN, & JESSE B. WRIGHT, "An analysis of a logical machine using parenthesis-free notation," *MTAC*, v. 8, 1954, p. 53-57.

## Tables for the Determination of Fundamental Solutions of Equations in the Theory of Compressible Fluids

1. **The pseudo-logarithmic plane.** It has been shown<sup>1,2,3</sup> that it is useful, when applying the hodograph method,<sup>4</sup> to consider the stream function  $\psi$  and the potential function  $\phi$  of compressible fluids in the so-called pseudo-logarithmic plane. In the case when the pressure density equation is  $p = \sigma \rho^\gamma$ ,  $\sigma$ ,  $\gamma$  being constants, and for subsonic flows, the cartesian coordinates of the pseudo-logarithmic plane are

$$(1) \quad \lambda = \frac{1}{2} \log \left[ \frac{1-T}{1+T} \left( \frac{1+hT}{1-hT} \right)^{1/h} \right], \quad T = (1-M^2)^{1/2}, \quad h = \left( \frac{\gamma-1}{\gamma+1} \right)^{1/2},$$

and  $\theta$  where  $M$  is the Mach number and  $\theta$  is the angle which the velocity vector forms with the positive  $x$  direction of the physical plane (*i.e.*, the plane in which the flow actually takes place).<sup>5</sup>

The equations for the potential ( $\phi$ ) and the stream function ( $\psi$ ) assume, when considering the flow in the  $\lambda$ ,  $\theta$ -plane, the form

$$(2) \quad \phi_{\lambda\lambda} + \phi_{\theta\theta} + [I^3(I^{-1})_\lambda]\phi_\lambda = 0, \quad \phi_\lambda \equiv \frac{\partial \phi}{\partial \lambda}, \dots$$

$$(3) \quad \psi_{\lambda\lambda} + \psi_{\theta\theta} + [I^{-1}(I)_\lambda]\psi_\lambda = 0.$$

Here  $l \equiv l(\lambda) = \rho^{-2} (1 - M^2)$  is a known function<sup>6</sup> of  $\lambda$ . If we introduce so-called *modified stream and potential functions*

$$(4) \quad \psi^* = \psi/\kappa,$$

$$(5) \quad \phi^* = \kappa\phi,$$

$$(6) \quad \kappa = (1 - M^2)^{-1} (1 + \frac{1}{2}(\gamma - 1)M^2)^{-1/2(\gamma-1)}$$

then we obtain for  $\psi^*$  and  $\phi^*$  equations

$$(7) \quad \psi^*_{,\lambda\lambda} + \psi^*_{,\theta\theta} + 4F_1(\lambda)\psi^* = 0,$$

$$(8) \quad \phi^*_{,\lambda\lambda} + \phi^*_{,\theta\theta} + 4F_2\phi^* = 0$$

where<sup>7</sup>

$$(9) \quad F_n = (-1)^n \frac{(\gamma + 1)M^4}{64} \left[ \frac{a_n M^4 + 4(3 - 2\gamma)M^2 - 16}{(1 - M^2)^2} \right], \quad n = 1, 2,$$

$$a_1 = (3\gamma - 1), \quad a_2 = \gamma - 3.$$

**2. Singular solutions of type S.** By a fundamental solution of a linear differential equation we mean a function  $S^*(\lambda, \theta; \lambda_0, \theta_0)$  which for a fixed  $(\lambda_0, \theta_0)$  is a solution of the differential equation and which possesses a logarithmic singularity in the vicinity of  $(\lambda_0, \theta_0)$ . In the case of equations (2) and (3) singularities of this type (denoted as singularities of type S)<sup>8</sup> can be obtained by continuing the coefficients  $F_1$  and  $F_2$  to complex values of  $\lambda$  and then forming for  $\psi^*$  the expression

$$(10) \quad \frac{1}{2}A[\log(\lambda - \lambda_0)^2 + (\theta - \theta_0)^2] + B$$

where

$$(11) \quad A = H \left[ 1 - \int_{z_0}^z \int_{\bar{z}_0}^{\bar{z}} F_1 dZ_1 d\bar{Z}_1 + \int_{z_0}^z \int_{\bar{z}_0}^{\bar{z}} F_1 \left( \int_{z_0}^{z_1} \int_{\bar{z}_0}^{\bar{z}_1} F_1 dZ_2 d\bar{Z}_2 \right) dZ_1 d\bar{Z}_1 + \dots \right],$$

$$(12) \quad B = H \left[ \int_{z_0}^z \int_{\bar{z}_0}^{\bar{z}} G_1 dZ_1 d\bar{Z}_1 - \int_{z_0}^z \int_{\bar{z}_0}^{\bar{z}} F_1 \left( \int_{z_0}^{z_1} \int_{\bar{z}_0}^{\bar{z}_1} G dZ_2 d\bar{Z}_2 \right) dZ d\bar{Z}_1 + \dots \right],$$

$$G = -\frac{1}{\xi} \frac{\partial(H^{-1}A)}{\partial Z} - \frac{1}{\xi} \frac{\partial(H^{-1}A)}{\partial \bar{Z}}, \quad \xi = Z - Z_0, \quad \bar{\xi} = \bar{Z} - \bar{Z}_0,$$

$$Z = \lambda + i\theta, \quad \bar{Z} = \lambda - i\theta, \quad Z_0 = \lambda_0 + i\theta_0, \quad \bar{Z}_0 = \lambda_0 - i\theta_0.$$

We obtain the corresponding singularity  $\phi^*$  replacing in  $A$  and  $B$  functions  $H$  and  $F_1$  by  $H^{-1}$  and  $F_2$ , respectively.<sup>9</sup>

In order to evaluate numerically the integrals in (11) and (12) and corresponding expressions for  $\phi^*$  we need the tables of  $\text{Re}(F_s)$ ,  $\text{Im}(F_s)$ ,  $\kappa = 1, 2$  for complex values of the arguments  $\lambda$ .

Re( $\lambda$ )	Im( $\lambda$ )	Re( $T$ )	Im( $T$ )	Re( $F_1$ )	Im( $F_1$ )	Re( $F_2$ )	Im( $F_2$ )
$-\infty$	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.1513	0.0000	.9750	0.0000	.0017	0.0000	-.0068	.0000
	.0654	.9752	-.0034	.0016	.0005	-.0066	-.0019
	.1309	.9759	-.0069	.0014	.0009	-.0059	-.0038
	.1963	.9772	-.0100	.0010	.0013	-.0043	-.0051
	.2617	.9788	-.0130	.0007	.0015	-.0026	-.0061
	.3271	.9808	-.0157	.0003	.0016	-.0008	-.0064
	.3925	.9833	-.0180	-.0003	.0015	+.0010	-.0062
	.4579	.9859	-.0200	-.0007	.0014	.0026	-.0056
	.5233	.9888	-.0216	-.0011	.0011	.0040	-.0045
	.5887	.9918	-.0228	-.0012	.0008	.0049	-.0032
	.6541	.9949	-.0235	-.0014	.0005	.0054	-.0017
	.7195	.9980	-.0238	-.0014	.0001	.0055	-.0003
	.7849	.9989	-.0237	-.0013	-.0001	.0054	.0001
-0.8047	.0000	.9473	.0000	.0087	.0000	-.0348	.0000
	.0654	.9478	-.0076	.0081	.0027	-.0329	-.0112
	.1309	.9496	-.0151	.0068	.0052	-.0270	-.0210
	.1963	.9526	-.0219	.0046	.0069	-.0185	-.0276
	.2617	.9565	-.0282	.0022	.0077	-.0087	-.0308
	.3271	.9613	-.0339	-.0003	.0077	.0013	-.0307
	.3925	.9667	-.0385	-.0024	.0069	.0098	-.0276
	.4579	.9726	-.0422	-.0041	.0056	.0164	-.0223
	.5233	.9789	-.0450	-.0053	.0040	.0208	-.0158
	.5887	.9852	-.0469	-.0059	.0023	.0231	-.0091
	.6541	.9916	-.0479	-.0058	.0006	.0233	-.0026
	.7195	.9981	-.0479	-.0055	-.0009	.0218	.0034
	.7849	1.0042	-.0473	-.0049	-.0021	.0193	.0083
-0.6020	.0000	.9161	.0000	.0259	.0000	-.1044	.0000
	.0654	.9172	-.0129	.0238	.0095	-.0961	-.0383
	.1309	.9206	-.0252	.0181	.0169	-.0732	-.0684
	.1963	.9261	-.0366	.0106	.0212	-.0416	-.0854
	.2617	.9332	-.0466	.0025	.0221	-.0092	-.0885
	.3271	.9416	-.0551	-.0045	.0200	.0190	-.0802
	.3925	.9510	-.0618	-.0098	.0162	.0396	-.0643
	.4579	.9608	-.0669	-.0130	.0115	.0523	-.0454
	.5233	.9709	-.0703	-.0143	.0066	.0575	-.0262
	.5887	.9810	-.0721	-.0143	.0022	.0570	-.0088
	.6541	.9909	-.0727	-.0134	-.0015	.0527	.0059
	.7195	1.0004	-.0719	-.0115	-.0043	.0456	.0173
	.7849	1.0094	-.0701	-.0093	.0064	.0371	.0257
-0.5281	.0000	.9000	.0000	.0401	.0000	-.1621	0.0000
-0.4582	.0000	.8810	.0000	.0629	.0000	-.2556	.0000
	.04	.88181	-.0120	.06018	.01654	-.2441	-.0679
	.0654	.8830	-.0195	.0558	.0261	-.2266	-.1072
	.1309	.8888	-.0380	.0381	.0445	-.1521	-.1818
	.1963	.8980	-.0545	.0158	.0515	-.0608	-.2081
	.20	.89856	-.05522	.01459	.05137	-.0564	-.2079
-0.4582	.2617	.9094	-.0684	-.0041	.0483	.0194	-.1939
	.3271	.9227	-.0794	-.0184	.0388	.0757	-.1541
	.3925	.9368	-.0877	-.0265	.0272	.1069	-.1064
	.4579	.9511	-.0932	-.0293	.0156	.1172	-.0612
	.5233	.9653	-.0967	-.0289	.0060	.1146	-.0226
	.5887	.9791	-.0978	-.0260	-.0018	.1030	.0072
	.6541	.9923	-.0973	-.0221	-.0073	.0874	.0292
	.7195	1.0048	-.0953	-.0176	-.0112	.0701	.0443
	.7849	1.0163	-.0919	-.0133	-.0135	.0529	.0535

$(F_2)$	$Re(\lambda)$	$Im(\lambda)$	$Re(T)$	$Im(T)$	$Re(F_1)$	$Im(F_1)$	$Re(F_2)$	$Im(F_2)$
.0000	-0.3466	.0000	.8403	.0000	.1425	.0000	-.5853	.0000
.0000		.04	.84168	-.01751	.12942	.04434	-.5462	-.1849
.0019		.0654	.8436	-.0285	.1195	.0689	-.4881	-.2877
.0038		.10	.84825	-.0426	.09286	.09366	-.3746	-.3879
.0051		.1309	.8535	-.0547	.0674	.1070	-.2591	-.4416
.0061		.17	.86179	-.06835	.03083	.11160	-.1167	-.4561
.0064		.1963	.8682	-.0769	.0095	.1087	-.0294	-.4411
.0062		.2617	.8860	-.0945	-.0295	.0877	.1274	-.3491
.0056		.3271	.9050	-.1075	-.0496	.0596	.2044	-.2341
.0045		.3925	.9247	-.1163	-.0556	.0337	.2241	-.1293
.0032		.4579	.9438	-.1215	-.0534	.0133	.2119	-.0494
.0017		.5233	.9625	-.1240	-.0468	-.0016	.1845	.0088
.0003		.5887	.9799	-.1238	-.0385	-.0114	.1517	.0466
.0001		.6541	.9961	-.1215	-.0304	-.0177	.1189	.0699
.0000		.7195	1.0110	-.1175	-.0224	-.0212	.0887	.0829
.0112		.7849	1.0246	-.1123	-.0157	-.0227	.0620	.0889
.0210	-0.2682	.0000	.8000	.0000	.2837	.0000	-1.181	.0000
.0276	-.2562	.3275	.8889	-.1390	-.1050	.0718	.4285	-.2722
.0308	-.2560	.5237	.9625	-.1515	-.0643	-.0179	.2507	.0743
.0307	-.2559	.7193	1.0188	-.1388	-.0248	-.0334	.0970	.1298
.0276		.1968	.8375	-.1049	-.0365	.2015	.1749	-.8151
.0223	-.2558	.5889	.9829	-.1491	-.0991	-.0272	.1907	.1075
.0158	-.2557	.0000	.7921	.0000	.3217	.0000	-1.342	.0000
.0091		.0653	.7980	-.0408	.2413	.1821	-.9944	-.7737
.0026		.2623	.8633	-.1255	-.0923	.1312	.3900	-.5160
.0034		.3926	.9155	-.1472	-.0966	.0270	.3846	-.0952
.0083	-.2554	.0256	.7930	-.0163	.30819	.08095	-1.284	-0.3456
.0000		.0502	.7956	-.03165	.27223	.14881	-1.127	-.6331
.0383		.09987	.8056	-.06055	.15893	.23065	-.6380	-.9700
.0684		.14	.81717	-.08116	.06324	.24157	-.2327	-1.002
.0854	-.2552	.1313	.8144	-.0768	.0835	.2427	-.3171	-1.010
.0885		.6540	1.0018	-.1450	-.0359	-.0317	.1397	.1247
.0802	-.2550	.4578	.9398	-.1510	-.0813	-.0010	.3184	.0113
.0643	-.2549	.7852	1.0343	-.1316	-.0156	-.0330	.0618	.1289
.0454	-.2173	.3329	.8863	-.1563	-.1371	.0607	.5543	-.2191
.0262	-.1815	.5791	.9848	-.1737	-.0577	-.0460	.2211	.1807
.0088	-.1813	.0000	.7330	.0000	.7736	.0000	-3.307	.0000
.0059	-.1806	.0648	.7434	-.0590	.4632	-.5113	-1.908	-2.230
.0173	-.1792	.6544	1.0095	-.1670	-.0372	-.0479	.1434	.1864
.0257	-.1787	.3272	.8780	-.1739	-.1785	.0518	.7174	-.1713
.0000	-.1785	.2624	.8434	-.1620	.0022	.0077	.8638	-.5537
.0000	-.1783	.0000	.7223	.0000	.78138	.0000	-3.862	.0000
.0679		.0242	.7339	-.02275	.72795	.24278	-3.100	-1.068
.1072		.04	.73691	-.03669	.64325	.36818	-2.718	-1.614
.1818		.05006	.7390	-.0462	.57508	.44169	-2.412	-1.929
.2081		.10	.75633	-.08624	.19955	.56744	-.7558	-2.422
.2079		.10026	.7569	-.0868	.19337	.56525	-.7292	-2.411
.1939		.2192	.8198	-.1494	-.20073	.2465	.8695	-.9593
.1541		.5893	.9886	-.1738	-.0544	-.0474	.2093	.1855
.1064	-.1781	.7197	1.0279	-.1587	-.0236	-.0465	.0929	.1792
.0612	-.1780	.7856	1.0449	-.1493	-.0128	-.0434	.0523	.1684
.0226	-.1779	.3921	.9097	-.1800	-.1414	-.0017	.5529	.0283
.0072		.4576	.9388	-.1813	-.1064	-.0293	.4096	.1268
.0292	-.1778	.1968	.8072	-.1414	-.1816	.3108	.8132	-1.229
.0443	-.1777	.1307	.7709	-.1074	.0116	.5161	.0452	-2.161
.0535		.5244	.9655	-.1792	-.0771	-.0426	.2951	.1722

Re( $\lambda$ )	Im( $\lambda$ )	Re( $T$ )	Im( $T$ )	Re( $F_1$ )	Im( $F_1$ )	Re( $F_2$ )	Im( $F_2$ )
-.1644	.1739	.7878	-.1389	-.1904	.4119	.8786	-1.659
-.1473	.0000	.7000	.0000	1.2236	.0000	-5.309	.0000
-.1160	.0000	.6570	.0000	2.1870	.0000	-9.677	.0000
-.1142	.3268	.8711	-.2100	-.2491	-.0087	.9739	.0952
-.1136	.5911	1.0176	-.1875	-.0347	-.0645	.1333	.2470
-.1127	.2642	.8304	-.2031	-.3484	.0849	1.416	-.2377
-.1123	.0629	.6762	-.0867	.7043	1.5800	-2.802	-7.041
-.1116	.04073	.66325	-.0589	1.43653	1.4304	-6.153	-6.487
	.05	.66858	-.06762	1.17971	1.48423	-5.371	-6.113
	.06	.67341	-.08015	.86799	1.56464	-3.538	-7.002
	.0798	.6870	-.1062	.25015	1.50623	-.7528	-6.596
	.15819	.7496	-.1676	-.48526	.60628	2.214	-2.382
-.1112	.3902	.9072	-.2134	-.1767	-.0529	.6739	.2390
-.1108	.5883	.9964	-.1963	-.0537	-.0688	.2025	.2650
	.7172	1.0376	-.1779	-.0198	-.0595	.0785	.2281
-.1101	.5261	.9717	-.2056	-.0805	-.0725	.3015	.2850
-.1095	.4572	.9413	-.2120	-.1210	-.0703	.4645	.2938
-.1094	.1982	.7807	-.1874	-.4784	.3192	2.058	-1.160
-.1092	.1294	.7243	-.1516	-.4305	.9191	2.099	-3.769
-.1087	.7854	1.0564	-.1664	-.0078	-.0536	.0344	-.2065
-.1061	.0906	.6894	-.1216	-.0455	1.4992	.5786	-6.479
-.0807	.0000	.6000	.0000	.4668	.0000	-21.26	.0000
-.0628	.0468	.5909	-.1042	1.0202	5.0430	-3.409	-23.27
-.0613	.3248	.8680	-.2443	-.2913	-.0969	1.098	.4646
-.0610	.6497	1.0255	-.2056	-.0293	-.0791	.1134	.2999
-.0607	.0000	.5549	.0000	8.5619	.0000	-39.90	.0000
-.0590	.5957	1.0062	-.2159	-.0461	-.0879	.1732	.3337
-.0572	.2700	.8270	-.2454	-.4358	-.0673	1.694	.4255
-.0554	.0577	.5913	-.1321	-.7605	4.9886	5.036	-22.47
-.0528	.3838	.9062	-.2472	-.1961	-.1203	.7184	.5060
-.0527	.03085	.5521	-.0828	3.78079	7.85575	-16.07	-36.92
	.04	.5632	-.1030	1.4420	7.2060	-4.884	-33.74
-.0506	.04936	.5756	-.1213	-.16725	6.18058	2.574	-28.37
-.0499	.7115	1.0471	-.1967	-.0138	-.0723	.0567	.2743
-.0499	.5312	.9821	-.2311	-.0719	-.1034	.2627	.3958
-.0460	.2024	.7650	-.2459	-.7883	.0236	3.215	.2607
-.0457	.1260	.6769	-.2170	-1.6765	.8002	7.586	-2.639
-.0444	.4555	.9469	-.2442	-.1203	-.1201	.4342	.4732
-.0439	.7850	1.0692	-.1831	-.0006	-.0629	.0085	.2414
-.0423	.0000	.5000	.0000	.1836	.0000	-88.02	.0000
-.0407	.3229	.8670	-.2586	-.3007	-.1398	1.111	.6361
-.0391	.0000	.4889	.0000	21.5359	.0000	-103.8	.0000
-.0385	.6966	1.0287	-.2137	-.0268	-.0854	.1028	.3228
-.0365	.6001	1.0122	-.2236	-.0407	-.0950	.1513	.3584
-.0349	.0233	.4924	-.0868	.1402	.1818	-26.78	-88.88
-.0338	.2757	.8307	-.2648	-.4330	-.1516	1.632	.7649
-.0301	.0487	.5392	-.1655	-5.3652	7.5268	28.27	-32.98
-.0275	.1201	.6578	-.2394	-2.2320	.4906	9.923	-.8918
-.0256	.0000	.4309	.0000	51.3924	.0000	-255.0	.0000
	.00192	.4312	-.0098	50.4590	7.79669	-250.3	-39.89
	.01358	.4438	-.0655	21.74356	33.62862	-101.9	-169.4
	.02930	.4773	-.1227	-4.06835	21.9170	26.47	-105.1
-.0254	.05023	.5265	-.1714	-7.00589	8.07236	36.54	-35.04
	.06358	.5557	-.1926	-5.77547	4.41078	28.88	-17.62
	.10204	.6280	-.2309	-2.98574	.92169	13.68	-2.415
-.0254	.3774	.9047	-.2647	-.2005	-.1587	.7174	.6559
-.0236	.1224	.6546	-.2622	-2.3558	-.0276	10.15	1.591

$(F_2)$	$\text{Re}(A)$	$\text{Im}(A)$	$\text{Re}(T)$	$\text{Im}(T)$	$\text{Re}(F_1)$	$\text{Im}(F_1)$	$\text{Re}(F_2)$	$\text{Im}(F_2)$
.659	-.0231	.0691	.5638	-.2052	-.5637	3.2184	27.62	-11.91
.0000								
.0000								
.0952	-.0220	.7055	1.0514	-.2067	-.0103	-.0790	.0441	.2980
.2470	-.0201	.0000	.4000	.0000	844.3000	.0000	425.2	.0000
.2377	-.0186	.1943	.7518	-.2736	-.9256	-.1897	3.651	1.261
.041	-.0172	.5367	.9901	-.2436	-.0622	-.1173	.2228	.4440
.487	-.0168	.0108	.3939	-.0695	.3536	.7888	165.4	-404.9
.113	-.0151	.0367	.4698	-.1710	-.1631	.1449	85.83	-63.54
.002	-.0114	.4538	.9511	-.2617	-.1125	-.1477	.3944	.5708
.596	-.0103	.7851	1.0768	-.1919	.0043	-.0674	-.0086	.2580
.382								
	-.0072	.0175	.3759	-.1368	-.5972	.7007	328.2	-339.0
	-.00225	.0000	.1988	.0000	6885.0000	.0000	37290.	.0000
.2390		.07135	.5483	-.2550	-7.03253	-.39862	32.28	6.548
.2650		.12327	.6520	-.28075	-2.36513	-.46519	9.866	3.587
.2281		.19557	.7516	-.29248	-.91111	-.35304	3.467	1.948
.2850								
.2938		.26137	.8190	-.29218	-.47980	-.27753	1.733	1.302
		.32557	.8718	-.28652	-.28247	-.22472	.9905	.9555
.160		.38922	.9153	-.27770	-.17482	-.18566	.6056	.7419
.769		.45699	.9544	-.26600	-.10642	-.15368	.3697	.5904
.2065								
.479	-0.00225	.52564	.9881	-.25254	-.06310	-.12835	.2232	.4829
.0000		.58995	1.01533	-.23890	-.03620	-.10919	.1327	.4081
		.64888	1.03721	-.22580	-.01886	-.09458	.0736	.3540
.27		.72008	1.060335	-.20940	-.00682	-.07979	.0242	.3070
.4646		.78546	1.07880	-.19395	-.00561	-.06838	-.0132	.2616
.2999								
.0000		.0401	.4560	-.22137	-21.8887	1.16194	108.3	6.996
.3337		.0060	.2600	-.10704	-638.0050	549.7400	3535.	-2818.
	0.0000	.0000	.0000	.0000	$\infty$	$\infty$	$\infty$	.0000
.4255		.0248	.38623	-.2030	-57.2950	-1.7443	292.2	38.01
.47		.0751	.5435	-.25965	-7.2871	-.84843	33.23	8.931
.5060								
.92		.1238	.6522	-.2846	-2.3708	-.54963	9.684	3.956
.74		.1967	.7528	-.29514	-.89654	-.37663	3.389	2.023
		.2612	.8190	-.29424	-.47812	-.28723	1.719	1.340
.37		.3260	.8724	-.28813	-.27990	-.22929	.9775	.9714
.2743		.3888	.9156	-.27913	-.17371	-.18860	.5998	.7523
.3958								
.2607		.4622	.95759	-.2662	-.10146	-.15322	-.4153	.5435
.639		.5251	.9883	-.25368	-.06264	-.12966	.2212	.4875
		.5899	1.0158	-.2398	-.03561	-.10998	.1304	.4107
.4732		.6491	1.0378	-.22653	-.01829	-.09509	.0716	.3558
.2414		.7193	1.0608	-.21012	-.00360	-.08023	.0204	.3029
.0000								
.6361		.7857	1.0794	-.1945	.00605	-.06863	-.0146	.2626
.0000	.0019	.0187	.3464	-.2000	-.9771	-.2243	285.6	93.65
	.0062	.0371	.4330	-.2500	-.2369	-.9431	112.0	62.84
	.0163	.0644	.5196	-.3000	-.7018	-4.6774	29.15	27.79
.3228	.0188	.2894	.8457	-.3078	-.3533	-.3165	1.200	1.362
.3584								
.88	.0366	.1012	.6062	-.3500	-2.2466	-2.5925	7.405	13.43
.7649	.0722	.1447	.6928	-.4000	-.6400	-1.5496	1.066	6.842
.98	.0984	.3893	.9397	-.3420	-.0772	-.2886	.1881	1.053
	.1273	.1894	.7794	-.4500	-.0350	-.9719	-.7788	3.576
.8918	.2022	.2275	.8660	-.5000	.2100	-.6235	-1.200	1.871

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1 S. BERGMAN, "A formula for the stream function of certain flows," *Nat. Acad. Sci., Proc.*, v. 29, 1943, p. 276-281.2 S. BERGMAN, "On two-dimensional flows of compressible fluids," *Nat. Adv. Com. for Aeronautics*, Technical Note No. 972, 1945.

<sup>1</sup> S. BERGMAN, "Two-dimensional subsonic flows of a compressible fluid and their singularities," Amer. Math. Soc., *Trans.*, v. 62, 1947, p. 452-498.

<sup>2</sup> S. A. CHAPLYGIN, "On gas jets," *Scientific Memoirs*, Moscow Univ., Math. Phys. Sec., 21 (1902), p. 1-21. (English translation published by Nat. Adv. Com. for Aeronautics, Technical Note No. 1064, 1944, and also by Brown University, 1944.)

<sup>3</sup> See page 462 of the reference of note 3.

<sup>4</sup> See the reference in note 2 and equations (2.8) and (2.9) of the reference in note 3.

<sup>5</sup> See page 465 of the reference of note 3, where symbols  $P$  and  $P$  were used for the  $F_1$  and  $F_2$  of the present paper.

<sup>6</sup> See page 472 of the reference of note 3.

<sup>7</sup> We note that when we are carrying out the integrations in (11) and (12),  $\lambda_n$  are continued to complex values  $\lambda_n + i\Delta_n$ , and  $Z_n = \lambda_n + i\Delta_n$  and  $\bar{Z}_n = \lambda_n - i\Delta_n$  become two independent variables. Also see page 473 of the reference in note 3.

## Analysis of Problem Codes on the Maniac

The Los Alamos computer, the MANIAC, has solved problems of wide variety during the last few years. The mathematical structure of such problems has ranged from differential equations (particularly partial differential equations), integral equations, stochastic processes, purely algebraic problems to some in the domain of mathematical logics.

There are several reasons why a frequency analysis of the computer as used in several typical problems might be useful. From such a study one may learn of significant variations in such distributions from one type of problem to another. Further, one may reach conclusions about the selection of the computer vocabulary. Most importantly, however, one may use the quantitative results as guiding principles in the design of a new computer. The economy of computer design is connected with the question, "Is the desirability of a particular order<sup>1</sup> commensurate with the associated electronic hardware?" Finally, a frequency analysis enables one to form accurate estimates for the "running time" of a problem; this information aids considerably in efficient scheduling of computer time. A knowledge of operation times for subroutines enables one to make rather good time estimates of lengthy problems during the formulation stage.

These frequency distributions can, of course, be gathered by hand. A more obvious way is to have the computer itself perform the analyses. A routine has been developed for this purpose and is called the "Code Analyzer."

The Code Analyzer gives the following information about a computer problem:

1. the frequency of occurrence of each order as it appears in the code—more briefly, a "static" count
2. the distribution in per cent of these static counts
3. the frequency of performance of each order during the running of a representative cycle of the problem—more briefly, a "dynamic" count
4. the distribution in per cent of these dynamic counts
5. the total time consumed performing each order of the vocabulary; i.e., (3) multiplied by the time needed to perform one such order
6. the per cent of the total time used by each order
7. the totals for (1), (3), (5).

The count in (1) is obtained by simply scanning linearly through the code of the problem and recording the occurrence of each order. The distribution (2) is

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the result of normalizing the distribution (1). To obtain (3), frequency of dynamic count, the problem must be run by means of an interpretive routine<sup>2</sup> (in our case it has actually the same vocabulary as the computer) which tallies one for the order as each instruction proper is performed. Distribution (4) is simply a normalization of (3). The operation times for the various individual orders are stored as constants within the Code Analyzer, and therefore only a simple multiplication is needed to obtain (5) in all cases except for the shift orders which are, of course, of variable duration. Each shift order is examined to determine the number of places shifted and the time computed accurately according to a simple linear expression.

Operation times were obtained by taking the time difference of a large number of passes through a control loop with and without each order. Dividing this time difference by the total number of passes, one obtains a realistic measure of the individual operation times that includes the access time for each instruction.

When the above statistical information has been gathered, the percentages and totals are calculated and all six columns (1 to 6, above) are printed, preceded by a column which lists the MANIAC's thirty-six orders. Totals (7) are printed below the appropriate columns. A brief explanation of the vocabulary symbols is given in Table 1.

Conceptually, the idea of the Code Analyzer is very simple. In practice, this would also be true provided a computer had only one medium of storage which was sufficiently capacious to contain *simultaneously* the Code Analyzer and the problem at hand. However, this condition does not obtain for the MANIAC which has one thousand and twenty-four locations of electrostatic storage (primary) and ten thousand locations of magnetic drum storage (secondary).

The communication between the two media gives rise to considerable complications during the interpretative process. As a result, the code for the Code Analyzer itself is by no means short; indeed, it has one thousand instructions, apart from numerical storage.

The procedure is as follows: First, the problem to be analyzed is loaded into the storage normally, as if it were going to be run. Then a key instruction is inserted at the end of a typical cycle to indicate the terminal point of interpretation. With very short problems, or with subroutines, this stop is the end of the problem or the exit from the subroutine. The code of the problem residing in the electrostatic memory is then transferred to the drum at a place that the Code Analyzer will use as a simulated one thousand twenty-four word memory during interpretation. In the interest of speed, the numerical storage, both constant and dynamic, remains in the true electrostatic memory at all times. This is possible if there are several Code Analyzers, each occupying a different section of the electrostatic memory, so one may be chosen which does not conflict with the numerical storage of the problem being analyzed. The next step is to load the selected Code Analyzer into the memory. To allow for problems with prodigious storage, all of the code for the Code Analyzer does not reside in the electrostatic memory at one time. The Code Analyzer divides naturally into two parts since the counting is independent of the totaling, computing of percentages, and printing of the output. Thus, only the code for the counting is loaded immediately into electrostatic storage. The second part of the code is temporarily stored on the magnetic

drum. Then, with the help of an auxiliary read tape which contains the initial and final addresses of the problem code, the address where interpretation is to begin, and the initial and final addresses of the problem storage, the Code Analyzer proceeds with its work and prints the results as described above. In addition both the dynamic and static counts for each analysis are punched on paper tape.

TABLE 1

*A Brief Description of the MANIAC Vocabulary Symbols*

The letters are tetrads of binary digits. A, B, ..., F correspond to the hexadecimal characters 10, 11, ..., 15. Associated with the pair of vocabulary symbols for each order are three tetrads specifying the memory location of the operand. There are two instructions per word; the occurrence of some orders in duplicate is necessary for reference to the left or right portions of the word.

Order No.	Vocabulary Symbol	Interpretation
1	AA	Add number (N) in memory (M) to cleared accumulator (A)
2	AB	Subtract N in M from cleared A
3	AC	Recall N on magnetic tape to quotient register (Q)
4	AD	Record N in Q on magnetic tape
5	AE	Add absolute value of N in M to cleared A
6	AF	Subtract absolute value of N in M from cleared A
7	BA	Add N in M to uncleared A
8	BB	Subtract N in M from uncleared A
9	BC	Recall 50 words (one track) from drum to M
10	BD	Record 50 words in M to drum
11	BE	Add absolute value of N to uncleared A
12	BF	Subtract absolute value of N from uncleared A
13	CA	Unconditional transfer of control
14	CB	Unconditional transfer of control
15	CC	Conditional transfer of control
16	CD	Conditional transfer of control
17	CE	High speed print (Synchroprinter)
18	CF	Punch paper tape
19	DA	Multiplication (Round Off)
20	DB	Multiplication (No round off)
21	DC	Replacement of N in A to M
22	DD	Division
23	DE	Left shift
24	DF	Add associated address to number in A
25	EA	Slow speed print (Teletype)
26	EB	Place N from M in Q
27	EC	Place N from Q in M
28	ED	Drop sign of number in A
29	EE	Right shift
30	EF	Place associated address in A
31	FA	Address replacement in M
32	FB	Address replacement in M
33	FC	Half word replacement in M
34	FD	Half word replacement in M
35	FE (not used)	
36	FF	N on paper tape input to M
37	800	Add N from Q to A

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These tapes provide the data in a convenient and flexible form to make cumulative analyses of various forms.

Results to date include analyses of some typical Laboratory problems. In Table 2 the analysis of a problem in hydrodynamics is shown. Analyses are also given in Tables 3 and 4 for a Monte Carlo problem and for one in mathematical logics, respectively. A major portion of the MANIAC's subroutines has been analyzed. The data for thirty-seven subroutines are shown summarized in Table 5. The running time for the analysis of the hydrodynamics problem was twenty-five

TABLE 2  
*Analysis of the Code for a Problem in Hydrodynamics*

Vocabulary Symbol	Static Count	Percentage of Static Count	Dynamic Count	Percentage of Dynamic Count	Time	Percentage of Time
AA	156	13.5	3499	12.3	314.9	5.4
AB	8	0.6	309	1.0	27.8	0.4
AC	0	0.0	0	0.0	0.0	0.0
AD	0	0.0	0	0.0	0.0	0.0
AE	0	0.0	0	0.0	0.0	0.0
AF	1	0.0	40	0.1	3.6	0.0
BA	93	8.0	2745	9.6	247.0	4.2
BB	68	5.8	2149	7.5	193.4	3.3
BC	0	0.0	0	0.0	0.0	0.0
BD	0	0.0	0	0.0	0.0	0.0
BE	0	0.0	0	0.0	0.0	0.0
BF	4	0.3	4	0.0	0.3	0.0
CA	34	2.9	332	1.1	16.6	0.2
CB	49	4.2	936	3.3	46.8	0.8
CC	13	1.1	447	1.5	20.1	0.3
CD	24	2.0	770	2.7	34.6	0.6
CE	0	0.0	0	0.0	0.0	0.0
CF	0	0.0	0	0.0	0.0	0.0
DA	93	8.0	2498	8.8	2592.9	45.1
DB	3	0.2	80	0.2	83.0	1.4
DC	157	13.6	4163	14.6	249.7	4.3
DD	49	4.2	1154	4.0	1197.8	20.8
DE	28	2.4	763	2.6	106.8	1.8
DF	6	0.5	12	0.0	0.8	0.0
EA	0	0.0	0	0.0	0.0	0.0
EB	118	10.2	3225	11.3	209.6	3.6
EC	104	9.0	2430	8.5	160.3	2.7
ED	13	1.1	37	0.1	1.4	0.0
EE	11	0.9	407	1.4	53.7	0.9
EF	19	1.6	86	0.3	6.4	0.1
FA	38	3.2	1101	3.8	88.0	1.5
FB	42	3.6	1063	3.7	85.0	1.4
FC	0	0.0	0	0.0	0.0	0.0
FD	0	0.0	0	0.0	0.0	0.0
FF	17	1.4	0	0.0	0.0	0.0
800	3	0.2	33	0.2	6.1	0.1
Totals	1151		28333		5747.4	

minutes, the Monte Carlo problem took seven minutes, and the logical problem took twenty minutes. Each subroutine analysis was a matter of seconds.

Examination of these preliminary results provides quantitative estimates to such matters as the effect on the running time for a problem if the operation time of some order were decreased by some factor. In other words, attention may be focussed where the "shoe pinches most." For example, a decrease of fifty per cent in the multiplication time alone would cut the running time by 30 per cent in some of the typical problems. We also see that some orders could have been

TABLE 3

*Analysis of the Code for a Monte Carlo Problem*

Vocabulary Symbol	Static Count	Percentage of Static Count	Dynamic Count	Percentage of Dynamic Count	Time	Percentage of Time
AA	37	18.2	2111	14.4	189.9	6.6
AB	2	0.9	112	0.7	10.0	0.3
AC	0	0.0	0	0.0	0.0	0.0
AD	0	0.0	0	0.0	0.0	0.0
AE	2	0.9	499	3.4	44.9	1.5
AF	0	0.0	0	0.0	0.0	0.0
BA	20	9.8	1326	9.1	119.3	4.1
BB	11	5.4	730	5.0	65.7	2.2
BC	0	0.0	0	0.0	0.0	0.0
BD	0	0.0	0	0.0	0.0	0.0
BE	0	0.0	0	0.0	0.0	0.0
BF	0	0.0	0	0.0	0.0	0.0
CA	4	1.9	89	0.6	4.4	0.1
CB	12	5.9	395	2.7	19.7	0.6
CC	3	1.4	231	1.5	10.3	0.3
CD	10	4.9	786	5.3	35.3	1.2
CE	0	0.0	0	0.0	0.0	0.0
CF	0	0.0	0	0.0	0.0	0.0
DA	3	1.4	909	6.2	943.5	32.9
DB	4	1.9	356	2.4	369.5	12.9
DC	38	18.7	2972	20.4	178.3	6.2
DD	2	0.9	499	3.4	517.9	18.0
DE	9	4.4	1085	7.4	146.4	5.1
DF	1	0.4	89	0.6	6.2	0.2
EA	0	0.4	0	0.0	0.0	0.0
EB	9	4.4	864	5.9	56.1	1.9
EC	7	3.4	454	3.1	29.9	1.0
ED	8	3.9	210	1.4	8.4	0.2
EE	4	1.9	356	2.4	67.6	2.3
EF	6	2.9	109	0.7	8.1	0.2
FA	4	1.9	26	0.1	2.0	0.0
FB	0	0.4	0	0.0	0.0	0.0
FC	1	0.4	89	0.6	7.1	0.2
FD	2	0.9	89	0.6	7.1	0.2
FF	0	0.0	0	0.0	0.0	0.0
800	2	0.9	178	1.2	13.1	0.4
Totals	201		14564		2861.8	

sacrificed without too much pain or regret. Examination of subroutine times has led to a re-evaluation of some of the methods used and in some cases has resulted in vast improvements.

It is planned to continue gathering statistics of problems computed on the MANIAC to establish a more quantitative basis for some of our ideas and conjectures.

TABLE 4

*Analysis of the Code for a Problem in Mathematical Logics*

Vocabulary Symbol	Static Count	Percentage of Static Count	Dynamic Count	Percentage of Dynamic Count	Time	Percentage of Time
AA	147	19.4	2025	16.5	182.2	2.1
AB	6	0.7	20	0.1	1.8	0.0
AC	0	0.0	0	0.0	0.0	0.0
AD	0	0.0	0	0.0	0.0	0.0
AE	2	0.2	4	0.0	0.3	0.0
AF	1	0.5	1	0.0	0.0	0.0
BA	70	9.2	1018	8.3	91.6	1.0
BB	30	3.9	412	3.3	37.0	0.4
BC	0	0.0	0	0.0	0.0	0.0
BD	0	0.0	0	0.0	0.0	0.0
BE	0	0.0	0	0.0	0.0	0.0
BF	0	0.0	0	0.0	0.0	0.0
CA	26	3.4	30	0.2	1.5	0.0
CB	29	3.8	84	0.6	4.2	0.0
CC	28	3.7	625	5.1	28.1	0.3
CD	35	4.6	1075	8.7	48.3	0.5
CE	1	0.1	180	1.4	453.6	5.2
CF	0	0.0	0	0.0	0.0	0.0
DA	0	0.0	0	0.0	0.0	0.0
DB	1	0.1	9	0.0	9.3	0.1
DC	90	11.9	1745	14.2	104.7	1.2
DD	0	0.0	0	0.0	0.0	0.0
DE	45	5.9	978	7.9	106.6	1.2
DF	27	3.5	71	0.5	4.9	0.0
EA	4	0.5	4	0.0	6647.6	77.1
EB	27	3.5	1031	8.4	67.0	0.7
EC	20	2.6	626	5.1	41.3	0.4
ED	12	5.1	12	0.0	0.4	0.0
EE	31	4.1	93	0.7	20.6	0.2
EF	13	1.7	450	3.6	33.7	0.3
FA	19	2.5	423	3.4	33.8	0.3
FB	19	2.5	825	6.7	66.0	0.7
FC	8	1.0	15	0.1	1.2	0.0
FD	10	1.3	30	0.2	2.4	0.0
FE	0	0.0	0	0.0	0.0	0.0
FF	5	0.6	15	0.1	600.0	6.9
800	19	2.5	435	3.5	32.1	0.3
Totals	725		12236		8621.0	

TABLE 5

*Analyses of the Codes for Some of the Subroutines, Summarized*

Vocabulary Symbol	Static Count	Percentage of Static Count	Dynamic Count	Percentage of Dynamic Count	Time	Percentage of Time
AA	450	16.7	5659	17.6	509.3	3.5
AB	21	0.7	123	0.3	11.0	0.0
AC	0	0.0	0	0.0	0.0	0.0
AD	0	0.0	0	0.0	0.0	0.0
AE	7	0.2	23	0.0	2.0	0.0
AF	10	0.3	10	0.0	0.9	0.0
BA	187	6.9	2643	8.2	237.8	1.6
BB	83	3.0	1833	5.7	164.9	1.1
BC	0	0.0	0	0.0	0.0	0.0
BD	0	0.0	0	0.0	0.0	0.0
BE	0	0.0	0	0.0	0.0	0.0
BF	3	0.1	38	0.1	3.4	0.0
CA	66	2.4	324	1.0	16.2	0.1
CB	68	2.5	476	1.4	23.8	0.1
CC	92	3.4	1572	4.9	70.7	0.4
CD	101	3.7	2601	8.1	117.0	0.8
CE	42	1.5	378	1.1	952.5	6.6
CF	0	0.0	0	0.0	0.0	0.0
DA	15	0.5	81	0.2	84.0	0.5
DB	15	0.5	227	0.7	235.6	1.6
DC	349	13.0	5079	15.8	304.7	2.1
DD	11	0.4	81	0.2	84.0	0.5
DE	136	5.0	2184	6.8	251.1	1.7
DF	70	2.6	429	1.3	30.0	0.2
EA	6	0.2	6	0.0	9971.4	69.8
EB	95	3.5	1634	5.1	106.2	0.7
EC	61	2.2	1045	3.2	68.9	0.4
ED	227	16.2	227	0.7	9.0	0.0
EE	54	2.0	255	0.7	33.1	0.2
EF	65	2.4	717	2.2	53.7	0.3
FA	53	1.9	906	2.8	72.4	0.5
FB	67	2.4	2214	6.9	177.1	1.2
FC	29	1.0	36	0.1	2.8	0.0
FD	27	1.0	46	0.1	3.6	0.0
FF	5	0.1	15	0.0	600.0	4.2
800	58	2.1	1147	3.5	84.8	0.5
Totals	2572		32009		14283.3	

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Work performed under the auspices of the Atomic Energy Commission.

<sup>1</sup> An "order" here refers to a single arithmetical or logical operation; e.g., add, multiply, transfer control. The ensemble of orders constitutes the computer vocabulary. An "instruction" refers to an order and its associated address.

<sup>2</sup> An interpretive routine is one which simulates a computer control and arithmetic unit. In effect, it translates and performs a coded sequence of instructions.

## TECHNICAL NOTES AND SHORT PAPERS

## Wolfram, Vega, and Thiele

In my article "New information concerning ISAAC WOLFRAM's life and calculations," *MTAC*, v. 4, 1950, p. 185-200, special consideration is given to his extraordinary table of  $\ln x$  to 48 D, as published in J. C. SCHULZE, *Recueil de Table Logarithmiques*, v. 1, 1778, p. 190-258. In this table are 3457 arguments, not 3462 as stated on p. 193, line -7, and p. 197, line 8. This erroneous statement was caused by overlooking the fact that in the 69 pages of the table there were ten groups of figures on every page *except* 256, where there were only nine groups. Thus this page reduced the estimated number of arguments by 5. Hence certain changes must be made in the text.

Following the change indicated above on p. 193, *for* 2230, *read* 2225; *for* 928, *read* 904; *for* 533, *read* 552. In addition to the necessary change indicated on p. 197 are others. First of all 9579 was a *misprint* for 9599. The reduction of the number of arguments in WOLFRAM by 5 means that there should be 79 arguments in THIELE, not 74, which are not in WOLFRAM; the additional arguments to the 74 listed are: 6049, 7453, 9707, 9821, 9877. In the last seven lines of p. 196, *for* 3456, *read* 3457; *for* 2280, *read* 2225; *for* 74, *read* 79.

In referring to VEGA's 1794 reprint of WOLFRAM's table, p. 194-195, I failed to note that VEGA gave only 3451 arguments, that is, 6 less than WOLFRAM. This fact was brought to my attention in June 1954, by Dr. ALAN FLETCHER, of the University of Liverpool. I now find that the 6 WOLFRAM arguments omitted in VEGA are the composite numbers 2215, 2225, 2233, 2299, 2387, 2401. Since no one of these omissions is a prime, PETERS' and STEIN's Table 13, based on VEGA, is unchanged.

Next, I refer to two matters in a letter of April 6, 1953, from my friend Mr. C. R. COSENS of the University of Cambridge. On p. 197, I had written concerning THIELE's 1908 table, "Curiously enough WOLFRAM's error in no. 28 (7853) is corrected. This is indeed a major mystery; the only explanation which I can offer is that the typesetter substituted an 8 for a 7, by mistake which THIELE did not observe!" The correction of this error in WOLFRAM was published by BURCKHARDT in 1817. I agree with COSENS that a better explanation of THIELE's achievement in this regard may have been that BURCKHARDT's correction had been brought to his attention.

The second matter which Mr. COSENS discusses at some length in his letter, has reference to the *Caliberstabe*, p. 189, lines 2-5, and footnote 34. Mr. COSENS shows that such a *Caliper Rule*, with scales, was used in connection with artillery; that the weight of a round shot equals the cube of its diameter. The diameter (bore of the gun) would be the cube root of the weight.

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## A Note on Approximating Polynomials for Trigonometric Functions

High speed automatic digital calculators have two means available for the evaluation of  $\sin x$  and  $\cos x$  when  $x$  is given. Either a table of values of the

required function may be held in the store<sup>1</sup> of the machine, and the given value obtained by interpolation; or the machine can calculate the required value from a number of terms of an infinite series. The former procedure is likely to be unsatisfactory if a high degree of accuracy is required since it is rarely possible to store enough function values to make linear, or even quadratic, interpolation feasible.

If function values are to be calculated directly from a series it is well known that the ordinary Taylor series is not the best possible. Chebyshev polynomials have the property that  $n$  terms of their series expansion of any function define a polynomial of degree  $n$  which minimizes the absolute value of the difference between any function value and the corresponding polynomial value in the interval  $(-1, +1)$ . Likewise, the Legendre polynomial expansion minimizes the integral square difference between function and approximating polynomial in  $(-1, +1)$ .

The two types of expansion find distinct applications in programming for a high speed computer.

(1) If the accuracy of the work is the maximum of which the machine is normally capable (often 9 or 10 decimal places) the Chebyshev series is appropriate, since it can give function values to the required precision with the minimum complexity.

(2) If the required accuracy is less than the full capacity of the machine (say 3-6 decimal places) and if the results of a number of calculations are to be combined additively as in summing a Fourier series, then the Legendre polynomial series will be the best basis of approximation.

In the work of this laboratory both of the above applications are of frequent occurrence, and since the numerical coefficients of the series required were apparently not to be found in the literature it is thought that they may be of interest to other workers in the field.

The Chebyshev polynomials are defined, following LANCZOS,<sup>2</sup> by:

$$T_0(x) = 1, \quad T_n(x) = \cos(n \cos^{-1} x),$$

and it may be shown that<sup>3</sup> the required expansions are:

$$\sin(\pi x/2) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\pi/2) T_{2n+1}(x),$$

$$\cos(\pi x/2) = J_0(\pi/2) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(\pi/2) T_{2n}(x).$$

The Legendre polynomials are conveniently defined by RODRIGUES' formula:

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

and it can be shown that:<sup>4</sup>

$$\begin{aligned} \sin(\pi x/2) &= 3 \cdot J_{3/2}(\pi/2) \cdot P_1(x) - 7 \cdot J_{7/2}(\pi/2) \cdot P_3(x) \\ &\quad + 11 \cdot J_{11/2}(\pi/2) \cdot P_5(x) - \dots, \\ \cos(\pi x/2) &= 1 \cdot J_{1/2}(\pi/2) \cdot P_0(x) - 5 \cdot J_{5/2}(\pi/2) \cdot P_2(x) \\ &\quad + 9 \cdot J_{9/2}(\pi/2) \cdot P_4(x) - \dots \end{aligned}$$

To make use of these expansions, for finding optimum polynomials for use with an automatic digital calculator, it is necessary to have numerical values for the functions  $J_n(\pi/2)J_{n+1/2}(\pi/2)$ . Since these do not appear to have been tabulated it was thought worth constructing the table given below. Values were obtained by means of the well-known series:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \left\{ 1 - \frac{(z/2)^2}{1!(\nu+1)} + \frac{(z/2)^4}{2!(\nu+1)(\nu+2)} - \dots \right\}$$

using the value ( $z = \pi/2$ ) taken to 20 decimal places. The resulting values were then rounded off to 11 decimal places and the resulting table checked by an application of the recursion formula:

$$J_\nu(z) = z/2\nu \{ J_{\nu+1}(z) + J_{\nu-1}(z) \}.$$

In addition, the value of  $J_{11.5}(\pi/2)$  was calculated directly from the recursion formula using the explicit value derived from the initial values:

$$J_{1/2}(\pi/2) = 2/\pi,$$

$$J_{3/2}(\pi/2) = 4/\pi^2,$$

and the table of values of  $\pi^{-n}$  to 25 decimal places computed by GLAISHER.<sup>5</sup>

$n$	$J_n(\pi/2)$	$n$	$J_n(\pi/2)$
0	0.47200 12157 7	.5	.63661 97723 7
1	0.56682 40889 1	1.5	.40528 47345 7
2	0.24970 16291 4	2.5	.13741 70540 3
3	0.06903 58882 9	3.5	.03212 73337 1
4	0.01399 60398 1	4.5	.00575 32170 8
5	0.00224 53571 2	5.5	.00083 61720 0
6	0.00029 83476 0	6.5	.00010 23428 0
7	0.00003 38506 4	7.5	.00001 08228 5
8	0.00000 33522 0	8.5	.00000 10077 8
9	0.00000 02945 7	9.5	.00000 00838 4
10	0.00000 00232 7	10.5	.00000 00063 0
11	0.00000 00016 7	11.5	.00000 00004 3
12	0.00000 00001 1	12.5	.00000 00000 3

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<sup>1</sup> A. D. BOOTH and K. H. V. BOOTH, *Automatic Digital Calculators*. Butterworths (London), 1953, p. 180.

<sup>2</sup> C. LANCZOS, *Tables of Chebyshev Polynomials*. NBS Applied Math. Ser. 9, Washington, D. C., 1952, p. V.

<sup>3</sup> G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. 2nd ed., Cambridge, 1944, p. 21.

<sup>4</sup> J. BAUER, *Crelle's Journal*, v. LVI, 1859, p. 113.

<sup>5</sup> J. W. L. GLAISHER, *London Math. Soc., Proc.*, v. 8, 1877, p. 140.

### Continued Fraction Expansion of 2<sup>1</sup>

The Institute for Advanced Study computer is being used to compute extensive continued fraction expansions of certain real algebraic numbers. The

interest lies in comparing statistics of such expansions with known distributions of these statistics over random numbers. For example, KHINTCHINE<sup>1</sup> has shown that over random numbers  $x$  uniformly distributed between 0 and 1, the sum  $S_n(x)$  of the first  $n$  partial quotients of  $x$  is equivalent in the sense of Bernoulli to  $Z_n = n \log n / \log 2$ .

The first result is the computation of more than 2000 partial quotients of  $2^1$ . The table below shows  $S_n(2^1)$  for  $n = 100(100)2000$ , with  $Z_n$  given for comparison. It appears that  $S_n(2^1)$  oscillates considerably in relation to  $Z_n$ , being most of the time larger, up to a factor of about 2. We do not know whether this deviation is significant, since<sup>1</sup> oscillations of  $S_n(x)$  of this type occur for almost all  $x$ . Expansions of additional numbers, as well as more detailed statistics, will follow.

The code depends on subroutines which do the necessary algebra on polynomials whose coefficients are  $p$ -tuples of computer words, for arbitrary and variable  $p$ . At present it handles cubic polynomials; a generalization to  $n$ th degree polynomials is planned. The methods and results will be reported later at greater length.

$n$	$n \log n / \log 2$	$S_n(2^1)$
100	664.4	1384
200	1528.8	2283
300	2468.6	2834
400	3457.5	3471
500	4482.9	4191
600	5537.3	12636
700	6615.8	18190
800	7715.1	18777
900	8832.4	19139
1000	9965.8	19724
1100	11113.6	20322
1200	12274.6	21825
1300	13447.6	22873
1400	14631.7	23293
1500	15826.1	24271
1600	17030.2	25259
1700	18243.2	25819
1800	19464.8	26442
1900	20694.4	27063
2000	21931.6	41198

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<sup>1</sup> A. KHINTCHINE, "Metrische Kettenbruchprobleme," *Compositio Mathematica*, v. 1, 1935, p. 361-382.

### The Values of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ and their Logarithms Accurate to 28 Decimals

The values of  $\Gamma(\frac{1}{3})$ ,  $\Gamma(\frac{2}{3})$ ,  $\log \Gamma(\frac{1}{3})$ ,  $\log \Gamma(\frac{2}{3})$  were computed to 28 decimals using the series

$$\log \Gamma(2+x) = C_1x + C_2x^2 - C_3x^3 + C_4x^4 - C_5x^5 + \dots + (-1)^r C_r x^r + \dots$$

where  $C_1 = 1 - \gamma$ ,  $\gamma = \lim \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$ : Euler's constant

$$C_r = \frac{1}{r} \left( \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \dots \right) \quad (r = 2, 3, \dots).$$

The values of  $S_r = \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \dots$  and  $\gamma$  were taken from Stieltjes' table.<sup>1</sup>

Part of the calculation was done with the assistance of Mr. E. V. HANKAM on an IBM (602-A type) calculating punch. Uhler's radix table was used for getting the antilog of  $\log \Gamma$ . The values  $\Gamma(\frac{1}{2})$  and  $\Gamma(\frac{3}{2})$  were required for calculating the power series coefficients of Bessel functions of order  $\frac{1}{2}$  and of functions related to them.

The values were checked by the identity

$$\sqrt{3}\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = 2\pi$$

$\Gamma(\frac{1}{2}) =$	2.67893	85347	07747	63365	56929	410
$\Gamma(\frac{3}{2}) =$	1.35411	79394	26400	41694	52880	282
$\log \Gamma(\frac{1}{2}) =$	.98542	06469	27767	06918	71740	370
$\log \Gamma(\frac{3}{2}) =$	.30315	02751	47523	56867	58628	174

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<sup>1</sup> H. T. DAVIS, *Tables of Higher Mathematical Functions*, v. II, The Principia Press, 1935, p. 244.

### Modification of a Method for Calculating Inverse Trigonometric Functions

The 605 programming that I gave recently<sup>1</sup> fails for arguments near  $2^{-1}$ . The reason for this failure is that the double angle formulations used multiply round-off errors until they are intolerably large. These formulations were originally introduced to assure that  $\cos 2\theta$  depend on both  $\sin \theta$  and  $\cos \theta$ . Upon closer examination it was found that it is only necessary that  $\cos 2\theta$  depend on  $\sin \theta$ , hence we may use

$$\cos 2\theta = 1 - 2 \sin^2 \theta.$$

The use of the above formula and

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

avoids the errors mentioned and is just as easily programmed for the 605.

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<sup>1</sup> RICHARD L. LA FARA, *MTAC*, v. 8, 1954, p. 132-139.

## REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

- 1[F].—A. GLODEN, *Table des factorisation des Nombres  $N^a + 1$  dans l'intervalle  $10000 < N \leq 20000$* . Manuscript of 84 leaves deposited in the UMT FILE.

This is an extension of earlier tables by the same author. Many of the entries have been completely factored. All unknown factors lie beyond 800 000. The author is preparing a table for the range  $20000 < N \leq 30000$ . [For previous tables of this kind see *MTAC*, v. 2, p. 211, 252, 300; v. 3, p. 21, 118–9, 486; v. 4, p. 224; v. 5, p. 28, 1334; v. 6, p. 102; v. 7, p. 33–4; v. 8, p. 166.]

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- 2[F].—RUDOLPH ONDREJKA, *List of the First 17 Perfect Numbers*. Two typewritten pages deposited in the UMT FILE.

Decimal values of the first seventeen perfect numbers, having the following respective numbers of digits: 1, 2, 3, 4, 8, 10, 12, 19, 37, 54, 65, 77, 314, 366, 770, 1327, 1373.

Computation of the first twelve perfect numbers was done by the author with the use of his table of  $2^n$ ,  $n = 1(2)411$ . The last five were computed by H. S. UHLER and were checked by the present author. The present list is believed by the author to be error-free.

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- 3[F].—R. J. PORTER, *A List of Groups and Series to serve for computations of Irregular Negative Determinants of Exponent  $3n$* . 274 typewritten pages deposited in the UMT FILE.

This is very closely related to the author's UMT 155 [*MTAC*, v. 7, p. 34] and UMT 185 [*MTAC*, v. 8, p. 96–7].

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- 4[K].—D. V. LINDLEY & J. C. P. MILLER, *Cambridge Elementary Statistical Tables*. Cambridge University Press, London, 1953, 35 p.,  $21.9 \times 27.9$  cm. Price \$1.00. (paper)

"This set of tables is concerned only with the commoner and more familiar and elementary of the many statistical functions and tests of significance now available.—The more familiar statistical tests are either based directly on the normal distribution or, in the case of the  $t$ ,  $\chi^2$  and  $F$  tests, they are derived therefrom. Percentage points for these tests are provided in the tables, mainly for significance levels 5%, 1%, and 0.1% in both one-sided and two-sided tests.—Tables of the more common transformations (of the data), square root, logarithm,

inverse circular and hyperbolic root-sines, together with that for the correlation coefficient, have been included." (From the Preface.)

A list of the tables follows:

- 1, 2. Cumulative normal  $\Phi(x)$  to 4D with differences for  $x = 0(.01)3(.1)4$  with normal ordinates to 4D for  $x = 0(.1)4$ , special values of  $x$  for  $\Phi(x) = .001(.001).03(.002).05(.05).50$  and several special values.
3. Percentiles 87.5, 95, 97.5, 99, 99.5, 99.9, 99.95 of the  $t$ -distribution for degrees of freedom 1(1)10, 12, 15, 24, 30, 40, 60, 120,  $\infty$  to 2D.
4.  $z = \tanh^{-1} r$  for  $r = 0(.02).8(.01).94(.001)1$ , to 3D with first differences. This is a transformation of the correlation coefficient.
5. Percentiles .5, 1, 2.5, 5, 90, 95, 97.5, 99, 99.5, 99.9 of the  $\chi^2$  distribution for degrees of freedom 1(1)30(10)100 to 3 or 4S.
6. Conversion of range to standard deviation for sample size  $n = 2(1)13$  to 4D. (Ratio of expected values.)
7. Percentiles 95, 97.5, 99, 99.9 of the  $F$ -distribution for degrees of freedom  $\nu_1 = 1(1)8, 10, 12, 24, \infty$  and degrees of freedom  $\nu_2 = 1(1)30(2)40, 60, 120, \infty$  to 3 or 4S (in general 2D).
8. 2000 random digits.
9. Various functions.  
For  $n = 0(1)100$ :  $n^2$ ,  $\sqrt{n}$  to 4D,  $1/n$  to 5D,  $1/\sqrt{n}$  to 5D.  
For  $x = 0(.01)1$ :  $\sin^{-1} \sqrt{x}$ ,  $\sinh^{-1} \sqrt{x}$ ,  $\sinh^{-1} \sqrt{10x}$ ,  $\sinh^{-1} \sqrt{100x}$  all to 3D with first differences.  
For  $x = 0(.01)10$ :  $x^2$  exact;  $\sqrt{x}$ ,  $\sqrt{10x}$ ,  $1/x$ ,  $1/\sqrt{x}$ ,  $1/\sqrt{10x}$  each to 4S when the first significant digit is  $> 2$  and to 5 figures when  $< 2$ , with first differences;  $\log x$  to 4D with first differences.  
For  $\log t = 0(.001)1$ :  $t$  to 4D for  $t < 2$  and to 3D for  $t > 2$  with first differences.
10.  $\log n!$  for  $n = 0(1)300$  to 4D.

The tables are arranged in a convenient format with notes on interpolation and asymptotic expressions for values beyond the given tables. A simple but not exact description of the tables is that this book is an abbreviated form of the FISHER & YATES Tables,<sup>1</sup> since that book contains, among other tables, most of those listed above. The principal area in which the book under review is more complete is in Table 7 and part of Table 9. Tables 3, 5, and 7 are based on tables from *Biometrika* but contain some additional values. Values which are reported here that do not appear either in the *Biometrika* tables or those of FISHER & YATES are those for  $\nu_2 = 32(2)38$  for all percentiles and those for  $\nu_1 = 7$  and 10 for percentile 99.9. A number of differences of a single unit in the final place were noted in the 99.9%  $F$ -table between the FISHER & YATES Tables<sup>1</sup> and the tables under review.

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<sup>1</sup> R. A. FISHER & FRANK YATES, *Statistical Tables for Biological, Agricultural, and Medical Research*. London & Edinburgh, 4th ed., 1953.

5[K].—H. G. ROMIG, *50-100 Binomial Tables*. New York, John Wiley & Sons, 1953, xxvii + 172 p., 18.7 × 22.4 cm., \$4.00.

These useful tables supplement the National Bureau of Standards Tables<sup>1</sup> which give individual terms and partial sums of terms of  $(q + p)^n$  to 6D for  $p = .01(.01)50$  and  $n = 2(1)49$ . Here the same quantities are given also to 6D for  $n = 50(5)100$ . The arrangement differs from that of the previous tables by giving all the entries for each  $n, p$  pair in adjacent columns, values of individual terms in one and cumulative sums in the other.

Dr. ROMIG provides very adequate explanation and illustrative examples to assure proper use of the *Tables*. The author also includes exact interpolation formulas determined by proper transformations of the interpolation formulas for the incomplete beta-functions. These formulas may be used for those probability determinations for intermediate  $p$  and  $n$  values for which ordinary linear interpolation is not satisfactory. In addition, there is a satisfying list of references which more completely cover the theory of interpolation.

These tables will find many uses in numerous areas of statistical analysis. Especially will they be useful in the area of statistical quality control where more refined evaluations of the binomial probabilities are necessary to determine the protection afforded by proposed sampling procedures in process control techniques as well as in acceptance plans.

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<sup>1</sup> NBSCL, *Tables of the Binomial Probability Distribution*. AMS no. 6, Washington, 1950. [MTAC, v. 4, p. 208-209.]

6[K].—W. L. STEVENS, "Tables of the angular transformation," *Biometrika*, v. 40, 1953, p. 70-73.

The following form of the angular transformation

$$\theta = 50 - \lambda \arcsin(1 - 2p), \quad \lambda = \sqrt{1000}$$

has been tabulated by the author in order to provide "a table similar in accuracy to that of the table of probits given by FISHER and YATES."<sup>1</sup> The author suggests that the present form of the transformation has the following advantages: (i)  $\theta$  ranges from 0.327 to 99.673 and therefore has almost the maximum possible accuracy for any given number of significant figures; (ii) the weight is given by the extremely simple expression  $n/1000$ , where  $n$  is the number of observations; (iii) like the probit function, complementary values of the function correspond to complementary values of the argument  $p = 50\%$  giving  $\theta = 50$ .

Three tables are presented, each to 3D. Table 1 gives  $\theta$  for  $100p = 0(0.1)50$  while for  $100p > 50$  entry is made for  $100(1 - p)$  and the transformed value is equal to 100 minus the tabular value. Proportional parts are given for linear interpolation. For small percentages Table 2 gives  $\theta$  for  $100p = 0(0.01)2.0$ , with proportional parts for linear interpolation when  $.05 < 100p < .2$ . For  $100p < 98$ ,  $\theta$  is determined as above by subtracting the tabular value obtained in Table 2 for  $100(1 - p)$  from 100. For  $100p < .05$  the formula  $\theta = 0.327 + 6.325\sqrt{p}$  may

be used. Table 3 provides the values for proper fractions whose denominations are less than or equal to 30.

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<sup>1</sup> R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*. London & Edinburgh, 1938.

7[K].—A. C. COHEN, JR. & JOHN WOODWARD, "Tables of Pearson-Lee-Fisher functions of singly truncated normal distributions," *Biometrics*, v. 9, 1953, p. 489-497.

A normal variable,  $x$ , with frequency function,

$$\phi(x) = (2\pi)^{-1/2} \sigma^{-1} \exp(x - m)^2 / 2\sigma^2,$$

for all real  $x$ , truncated (below) at  $x_0'$ , gives the truncated variable,  $x'$ , with frequency function,  $f(x') = \phi(x') / I_0(x')$ ,  $x_0' \leq x'$ , where  $I_0(x') = \int_{x_0'}^{\infty} \phi(x) dx$ . It is required to estimate  $m$  and  $\sigma$  on the basis of a sample of  $n$  observations on  $x = x' - x_0'$ .

Pearson and Lee<sup>1</sup> gave estimates based on the first two observed moments. FISHER<sup>2</sup> showed their estimates to be maximum likelihood estimates. The authors estimate  $\xi = \frac{x_0' - m}{\sigma}$  and  $\sigma$  by the relations (equivalent to the earlier ones)

$$(1) \quad \frac{n \sum x^2}{2(\sum x)^2} = \frac{1}{2} \left[ \frac{1}{z - \xi} \right] \left[ \frac{1}{z - \xi} - \xi \right] = g(\xi),$$

and

$$(2) \quad \sigma = \frac{\sum x}{n} \left[ \frac{1}{z - \xi} \right] = \frac{\sum x}{n} h(\xi),$$

where  $z = \phi(\xi) / I_0(\xi)$ . To facilitate computations they give tables of  $h(\xi)$  and  $g(\xi)$  to 8D except for the largest values of  $\xi$  (where 7D and 6D are given) for  $\xi = -4.(.1) - 2.5(.01).5(.1)3$ . The authors suggest using (1) to estimate  $\hat{\xi}$  and (2) with  $\hat{\xi}$  from (1) to estimate  $\hat{\sigma}$ .

The variances of the estimates and the correlation coefficient between the estimates are given by

$$\text{var}(\hat{\xi}) = \frac{\sigma^2}{n} \frac{1 - z(z - \xi)}{[1 - z(z - \xi)][2 - \xi(z - \xi)] - [z - \xi]^2} = \frac{\sigma^2}{n} W''(\xi),$$

$$\text{var}(\hat{\sigma}) = \frac{1}{n} \frac{2 - \xi(z - \xi)}{[1 - z(z - \xi)][2 - \xi(z - \xi)] - [z - \xi]^2} = \frac{1}{n} w'(\xi),$$

and

$$\rho(\hat{\xi}, \hat{\sigma}) = \frac{z - \xi}{\sqrt{[1 - z(z - \xi)][2 - \xi(z - \xi)]}}.$$

Tables of  $W'(\xi)$  and  $w'(\xi)$  are given to 6D and, of  $\rho(\xi, \sigma)$  to 4D for  $\xi = -3, (-1)2$ .

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<sup>1</sup> KARL PEARSON and ALICE LEE, "On the generalized probable error in multiple normal correlation," *Biometrika*, v. 6, 1908, p. 59-68.

<sup>2</sup> R. A. FISHER in *BAAS Math. Tables*, v. I, London, 1931, p. xxvi-xxxv.

8[K].—P. V. K. IYER & A. S. P. RAO, "Theory of the probability distribution of runs in a sequence of observations," *Indian Soc. Agricultural Stat.*, *Jn.*, v. 5, 1953, p. 29-77.

This paper investigates the theory of runs in which the succession of observations may be equal, ascending or descending. As such it augments the existing work on runs in which the equality assumption was not allowed. The purpose of the paper is to investigate the distribution of the number of ascending, descending and stationary runs in a sequence of  $n$  observations. Both the infinite case, where a particular value has a given probability of occurrence, and the finite case, where one knows the number of times a given value has occurred, is considered. For each of the three types of runs, the various related configurations are tabulated along with their probability of occurrence in the sequence and the number possible. No actual distributions are attempted in the paper. Variances and covariances for  $k$  kinds of elements with equal probabilities of occurrence are tabulated to 4D for  $K = 2(1)5$ , 10 and  $n = 30, 40, 50, 75, 100$  for ascending, stationary, and descending runs as well as for the total number of runs. The author also lists the algebraic expressions for the covariances of runs of lengths  $p$  and  $q$  for  $p = 1(1)4$ ,  $q = p(1)5$ , except for  $p = 4$  only  $q = 4$  is given; of runs of length  $p$  and  $q$  or more for  $p, q = 1(1)4$ ; and of runs of lengths  $p$  or more and  $q$  or more for  $p = 1, 2, 3, q = p + 1(1)4$ . For junctions the actual distributions are given for values of  $n = 4(1)7$ .

Mention is made of the possibility of using the results for testing randomness but the actual discussion of such applications are to be given in a separate article.

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9[K].—S. H. ABDEL-ATY, "Tables of generalized  $k$ -statistics," *Biometrika*, v. 41, 1954, p. 253-260.

The author gives a complete table of order 12 for expressing the sample  $k$ -statistics of TUKEY<sup>1</sup> in terms of the augmented monomial symmetric functions of DAVID & KENDALL<sup>2</sup> and vice versa. This is a very considerable extension of the table of WISHART,<sup>3</sup> which were through order 6, since results for lower orders are at once obtainable by a simple rule for the deletion of subscripts.

C. C. C.

<sup>1</sup> J. W. TUKEY, "Some sampling simplified," *Amer. Stat. Assn.*, *Jn.*, v. 45, 1950, p. 501-519.

<sup>2</sup> F. W. DAVID & M. G. KENDALL, "Tables of symmetric functions—part I," *Biometrika*, v. 36, 1949, p. 431-449. [*MTAC*, v. 4, p. 146.]

<sup>3</sup> J. WISHART, "Moment coefficients of the  $k$ -statistics in samples from a finite population," *Biometrika*, v. 39, 1952, p. 1-13. [*MTAC*, v. 7, p. 97.]

- 10[K].—J. H. CADWELL, "The statistical treatment of mean deviation," *Biometrika*, v. 41, 1954, p. 12-18.

The author wants to obtain properties of the mean deviation  $m$ , which are analogous to the well-known properties of the standard deviation  $\sigma$  of a normal distribution. To this end, the distribution of the quotient  $m/\sigma$  is approximated by the  $\chi^2$  distribution. He matches the first two moments of the two distributions with a small discrepancy for the third moment. Let  $\bar{m}(k, n)$  be the average of  $k$  mean deviations, each for samples of size  $n$  from a normal population with standard deviation  $\sigma$ . Then it is shown that  $c[\bar{m}(k, n)/\sigma]^{1.8}$  has approximately the  $\chi^2$  distribution with  $v$  degrees of freedom. Table 1 gives the values  $c$  to 4S and  $v$  to 1D, the expected values of  $\bar{m}(k, n)/\sigma$  to 4D and the variances of  $\bar{m}(1, n)/\sigma$  to 5D as functions of  $k$  and  $n$  for  $k = 1(1)10$ ,  $n = 4(1)10$ . Table 2 gives the same values for  $k = 1(1)5$ ,  $n = 10(5)50$ . Table 3 gives the lower and upper 2.5 percent and 5 percent points of the probability function of  $\bar{m}(k, n)$  for  $k = 1(1)10$  and  $n = 4(1)10$  to 3D. The values are exact for  $k = 1$ , and for other values the error will not exceed .003. For values of  $k$  beyond 10 a normal approximation can be used.

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- 11[K].—H. WEILER, "A new type of control chart limits for means, ranges, and sequential runs," *Amer. Stat. Assn., Jn.*, v. 49, 1954, p. 298-314.

In the usual theory of quality control charts the probability of a sample point falling outside the control limits, when the system is in control (a Type I error) is kept constant regardless of the number of items involved at each sample point. In the author's theory the average number of false alarms (Type I error) is a fixed percentage of the number of items tested, independent of the sample size. Given the variable  $X$ , normally distributed with known mean,  $m$ , and standard deviation,  $\sigma$ , let  $p$  be the probability that a random sample of  $n$  causes a false alarm at the upper limit, let  $a$  be the average number of articles tested before the false alarm is raised; then  $a = n/p$ . In Table I the author lists the values of  $B$  and  $B/\sqrt{n}$  to 2D, for  $a = 5000$ ,  $n = 3(1)6, 8, 10$ , since the upper control limit is given by  $m + B\sigma/\sqrt{n}$ . The lower control limit is given by  $m - B\sigma/\sqrt{n}$ . In Table II are listed the values of  $B/\sqrt{n}$  to 3S for  $n = 3(1)10(5)50$  and  $a = 1000(1000)5000$ . Suppose the population mean  $m$  changes to  $m + k\sigma$ ,  $k > 0$ . For a given  $n$  and  $B$ , the average number of items tested  $A(n)$  is a function of  $k$ .  $A(n)$  is plotted against  $k$  for  $a = 2000$ ,  $n = 5, 10, 20, 50$  (Chart I), and for  $a = 5000$ , same values of  $n$  (Chart II). From these charts the author concludes that if large values of  $k$  are expected,  $k > 1.6$  say, then small samples, e.g.,  $n = 5$ , should be used; while if small values of  $k$  are expected,  $k < 1$  say, large samples of  $n$ , e.g.,  $n = 10$  or 20, are more economical. In Chart III  $A(n)$  is plotted against  $k$  for fixed sample size  $n = 10$ ,  $a = 1000, 2000, 3000, 5000$ . This chart indicates that small values of  $a$  are useful only for the detection of small changes of the population mean.

Similarly for the control chart of the range in Table III are given to 2D the values of  $W_{1\sigma} \leq R \leq W_{2\sigma}$ , the control limits for the sample range  $R$ , for

$n = 3(1)10$ ,  $a = 1000(1000)5000$ . In Chart IV  $A(n)$  is graphed for  $n = 5, 10$ ,  $1.0 \leq k \leq 2.4$ , for the range. In Chart V  $A(n)$  is graphed for  $n = 10$ ,  $a = 1000, 2000, 5000$  for the range. For the range it is assumed that  $\sigma$  changes from  $\sigma$  to  $\sigma' = K\sigma$ ,  $K > 1$ .

The author next turns to the use of runs for controlling the mean, where  $\lambda$  = length of run. In Chart VI is graphed the value of  $A(n)$  for  $0 \leq k \leq 1.4$ ,  $n = 4, 10, 20$  and  $a = 4000$ ,  $\lambda = 2$ , and in Chart VII  $A(n)$  is graphed for the same values of  $n$  and  $a$  but  $\lambda = 3$ . In Chart VIII  $A(n)$  is graphed  $0 \leq k \leq 1.3$  for  $\lambda = 1$ ,  $n = 20$ , and for  $\lambda = 3$ ,  $n = 4$ . In the range  $0 < k \leq 1$ , the use of  $\lambda = 1$ ,  $n = 20$  is superior from the power sense to  $\lambda = 3$ ,  $n = 4$ .

The use of runs for the range charts reduces rather than improves the power of the chart and hence no charts are given. Considerable use is made of the author's two previous papers.<sup>1,2</sup>

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<sup>1</sup> H. WEILER, "On the most economical sample size for controlling the mean of a population," *Ann. Math. Stat.*, v. 23, 1953, p. 247-254.

<sup>2</sup> H. WEILER, "The use of runs to control the mean in quality control," *Amer. Stat. Assn., Jn.*, v. 48, 1953, p. 816-825.

12[K].—S. ROSENBAUM, "Tables for a nonparametric test of location," *Annals Math. Stat.*, v. 25, 1954, p. 146-150.

If two independent random samples of sizes  $n$  and  $m$  are drawn from a continuous statistical population, the probability that exactly  $i$  points of the sample of  $m$  will exceed the greatest value of the sample of  $n$  is:  $Q_i = n^{(m)}B(n+m-i, i+1)$ , where  $B$  is the complete Beta function. We can fix a probability level  $\epsilon$  and determine a value  $s_0$  such that:

$$\sum_{i=0}^{s_0-1} Q_i \leq \epsilon < \sum_{i=0}^{s_0} Q_i.$$

This paper presents tables of  $s = s_0 + 1$  for  $\epsilon = .99, .95$  and  $m, n = 1(1)50$ . These results can be used to test whether two samples came from the same population. The argument is identical if the number of values of the sample of  $m$  which are less than the smallest value of the sample of  $n$  is considered.

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13[K].—H. RUBEN, "On the moments of order statistics in samples from normal populations," *Biometrika*, v. 41, 1954, p. 200-227.

Many statisticians have treated the problem of evaluating the moments of order statistics in small samples drawn from normal populations. The results have been fragmentary, partly because no one has developed a systematic approach to the problem. The author gives a systematic approach in this paper. The method involves showing from a geometrical point of view that the moments of normal order statistics as well as the moment generating function of any order

statistic are closely related to the volumes of members of a class of hyperspherical simplices. These volumes involve calculation of a function  $u_p(x)$  (see the paper for details). Table 1 gives values of this function for  $x = 2(1)12$  and  $\beta = 1(1)49$  to 8 to 10D. These values are used to compute Table 2, which gives the first ten moments of the extreme members (smallest or largest values) in samples of size  $n$ ,  $1 \leq n \leq 50$  to 9 or 10S. Table 3 gives the second, third, and fourth moments about the mean of extremes in samples of size  $1 \leq n \leq 50$ , as well as the standard deviations, together with  $\beta_1$  and  $\beta_2 - 3$  to 7 or 8D.

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14[K].—T. J. TERPSTRA, "The exact probability distribution of the  $T$  statistic for testing against trend and its normal approximation," K. Ned. Akad. van Wetensch., *Proc.*, s. A, v. 56, 1953, p. 433-437.

Let  $x_{ih}$ ,  $i = 1, \dots, l$ ,  $h = 1, \dots, n_i$ , be  $l$  random samples of the random variables  $x_i$  which the null hypothesis states to have identical frequency distributions. Let  $\bar{U}_{ij}$ , WILCOXON's statistic,<sup>1</sup> be the number of pairs  $(h, k)$ ,  $h \leq n_i$ ,  $k \leq n_j$ , with  $i < j$  and  $x_{ih} < x_{jk}$ ;  $i, j = 1, \dots, l$ ,  $h, k = 1, \dots, n_i$ ; then  $T = \sum \sum_{i < j} \bar{U}_{ij}$ , a generalization of the  $T$  defined by MANN & WHITNEY,<sup>2</sup> was studied by the author<sup>3</sup> as a test against the alternate hypothesis of an upward term in the  $x_i$ . In the present paper, in Table 1 the author gives the exact distribution of  $T$  to 3D for  $n_1 \leq n_2 \leq n_3 \leq 5$ , and in Table 2 gives the .005, .01, .025, .05 and 0.1 significance levels for  $T$  (the smallest value of  $T$  for which the probability of its being exceeded is no greater than the relevant significance level) for the same values of the  $n$ 's. The values are also given for the normal approximation which are such that they indicate that for  $n_i \geq 5$  this approximation is good.

C. C. C.

<sup>1</sup> F. WILCOXON, "Individual comparisons by ranking methods," *Biometrics Bull.*, v. 1, 1945, p. 80-83.

<sup>2</sup> H. B. MANN & D. R. WHITNEY, "On a test of whether one of two random variables is stochastically larger than the other," *Ann. Math. Stat.*, v. 18, 1947, p. 50-60.

<sup>3</sup> T. J. TERPSTRA, "The asymptotic normality and consistency of Kendall's test against trend, when ties are present in a ranking," K. Ned. Akad. van Wetensch., *Proc.* s. A, v. 55, 1952, p. 327-333.

15[K].—H. O. HARTLEY & H. A. DAVID, "Universal bounds for mean range and extreme observation," *Annals Math. Stat.*, v. 25, 1954, p. 85-99.

The authors extend the theory of universal upper and lower bounds for  $E(w_n)$ , and universal upper bounds for  $E(x_n)$ , where  $x_n$  is the standardized extreme variate and  $w_n$  the standardized range of a sample of  $n$ . Table I gives the upper bound of  $E(x_n)$  to 4D for any population for  $n = 2(1)20$ . Also given for comparison are previously known values for symmetric populations. Table II gives the universal lower bound for  $E(w_n)$  over distributions with finite range  $-X \leq x \leq X$ . For  $n = 2(2)12$ , the bound is given to 3D for  $X = 1(1)5$ ; for  $n = 12(2)20$ , it is given to 3D for  $p = .95(.01).99$ , where  $p = X^2/(1 + X^2)$ . Specific values are easily computed from equation (58). As  $X \rightarrow \infty$ ,  $E(w_n) \rightarrow 0$  in agreement with previous results. It is shown that universal upper bounds

previously computed for the case of symmetric populations with infinite range are also applicable to the non-symmetric case, and to the case of finite range unless  $X < X_n$ , where  $X_n$  is the limit of the range for the population which maximizes  $E(w_n)$ . Values of  $X_n$  to 3D are tabulated for  $n = 2(1)20$ , and the algebraic form for the bound when  $X < X_n$  is given.

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- 16[K].—R. E. BECHHOFFER, "A single-sample multiple decision procedure for ranking means of normal populations with known variances," *Annals Math. Stat.*, v. 25, 1954, p. 16-39.

Let  $X_1, \dots, X_k$  be independent normal random variables of unit variance. Table I gives to 4D the value of  $d$  such that  $\gamma = \Pr\{X_1, \dots, X_{k-t} < X_{k-t+1} + d, \dots, X_k + d\}$  for  $\gamma = .05(.05).8(.02).9(.01).99, .999, .9995$  and for  $k = 2(1)10$ ,  $t = 1(1)[k/2]$  as well as 10 other pairs  $(k, t)$ . Table II gives to 4D the value of  $d$  such that  $\gamma = \Pr\{X_1 < X_2 + d < X_3 + 2d\}$  for  $\gamma = .2(.05).8(.02).9(.01).99$ . The tables enable one to compute the numbers of observations needed from normal populations of known variance in order to have confidence  $\gamma$  in certain statements about the order of the population means.

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- 17[K].—R. E. BECHHOFFER & MILTON SOBEL, "A single-sample multiple decision procedure for ranking variances of normal populations," *Annals Math. Stat.*, v. 25, 1954, p. 273-289.

Let  $U, V, W, X$  be independent chi-square random variables, each with  $n$  degrees of freedom. The paper provides 5D tables of  $\Pr\{U < \theta V\}$ ,  $\Pr\{U < \theta V, \theta W\}$ ,  $P\{U, V < \theta W\}$ ,  $\Pr\{U < \theta V < \theta^2 W\}$  and  $\Pr\{U < \theta V, \theta W, \theta X\}$ , for  $n = 1(1)20$  and  $\theta = 1.2(.2)2.2$ . The tables provide the confidence coefficients of certain statements about the order of the variances of normal populations.

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- 18[K].—C. W. DUNNETT & MILTON SOBEL, "A bivariate generalization of Student's  $t$ -distribution with tables for certain special cases," *Biometrika*, v. 41, 1954, p. 153-169.

The authors consider the simultaneous distribution of two variates,  $t_1 = z_1/s$  and  $t_2 = z_2/s$ . The  $z_i$  follow a normal bivariate distribution with zero means, the same variance  $\sigma^2$ , and correlation  $\rho$ . The variance  $\sigma^2$  is assumed independently estimated by  $s^2$  with  $n$  degrees of freedom. The probability integral is:

$$\text{Prob } \{t_1 \leq h; t_2 \leq h\} = P.$$

Tables of  $P$  and  $h$  are presented for  $n = 1(1)30(3)60(15)120, 150, 300, 600, \infty$  and  $\rho = .5$  and  $-.5$ .  $P$  is given to 5D for  $h = 0(.25)2.50$  and  $3.00$ , plus some addi-

tional values for larger  $h$  when  $n$  is small.  $h$  is given to 3D for  $P = .50, .75, .90, .95$  and  $.99$ .

An asymptotic expansion is derived for  $P$  and  $h$ ;

$$P = \sum_{i=0}^4 A_i/n^i \quad \text{and} \quad h = \sum_{i=0}^4 B_i/n^i.$$

Values of the  $A_i$  and  $B_i$  to 6D are presented for the same values of  $h$  and  $P$  mentioned above.

This distribution has applications in certain multiple decision problems.

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19[K].—J. M. SENGUPTA, "Significance level of  $\sum x^2/(\sum x)^2$  based on Student's distribution," *Sankhyā*, v. 12, 1953, p. 363.

In testing whether the mean of a normal population sampled is equal to a given value  $\mu$ , one ordinarily applies Student's " $t$ ," i.e.,  $t = \sqrt{n}(\bar{y} - \mu)/s$ , where  $\bar{y} = \frac{1}{N} \sum_{i=1}^N y_i$  and  $s^2 = \frac{1}{N-1} \sum_{i=1}^N (y_i - \bar{y})^2$  is the usual unbiased estimate of population variance. The author of the paper reviewed has noted that if one puts  $x_i = y_i - \mu$ , i.e., considers deviations about the hypothetical mean tested, then

$$(1) \quad \frac{\sum x^2}{(\sum x)^2} = \frac{t^2 + N - 1}{Nt^2}$$

and hence percentage points for the right-hand member of (1) can be obtained from percentage or significance levels of Student's " $t$ ." A table of such percentage points would therefore simplify computations for the test of a hypothetical mean and will in fact be very useful and time-saving when many  $t$ -tests are to be carried out. The presented table contains the 5% and 1% significance levels to 4D for  $t^2 + N - 1/Nt^2$  for sample sizes  $N = 2(1)30(10)60, \infty$ . The user of the table should

note that significant values of  $(t^2 + N - 1)/Nt^2$  are less than those tabulated.

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20[K].—J. C. SPITZ, "Matching in psychology," (Dutch, English Summary), *Statistica*, v. 7, 1953, p. 23-40.

Let  $p_r$  be the probability of exactly  $r$  matches in a random matching of two similar decks of 3 distinct cards. Then,  $p_0 = \frac{1}{6}$ ,  $p_1 = \frac{1}{2}$ ,  $p_2 = 0$ ,  $p_3 = \frac{1}{6}$ . In a series of  $n$  such random matchings, let  $R = r_1 + \dots + r_n$  be the total number of matches ( $0 \leq R \leq 3n$ ) and let  $P_{n,a} = \Pr(R \geq a)$ . Obviously,

$$(1) \quad P_{n,a} = \frac{1}{6}P_{n-1,a} + \frac{1}{2}P_{n-1,a-1} + \frac{1}{6}P_{n-1,a-2}$$

from which the value  $P_{n,a}$  is tabulated to 3D for  $n = 1(1)30$  and  $a = 0(1)3n$ . If in an actual experiment the number  $R = a$  of matches is such that  $P_{n,a} \leq .05$  (say), one rejects the hypothesis that the matchings were random.

For large  $n$ , by the central limit theorem,  $R$  is approximately normal. Now,  $R$  has a mean  $\mu = n$ , a variance  $\sigma^2 = n$  and a skewness  $\gamma_1 = 1/\sqrt{n}$ . Thus, for large  $n$ ,  $t = (R - \frac{1}{2} - n)/\sqrt{n}$  is about  $N(0, 1)$ . For  $n = 30$ , the resulting approximation to  $P_{n,a}$  appears to be fairly good. An even better approximation is obtained by replacing  $t$  by a type III variable with  $\mu = \sigma = 1$ ,  $\gamma_1 = 1/\sqrt{n}$ ; then  $P_{n,a}$  can be readily computed from SALVOSA's tables.<sup>1</sup>

The reviewer would expect a good approximation to  $P_{n,a}$  by replacing  $R$  by a Poisson variable with parameter  $n$  which has  $\mu = \sigma^2 = n$ ,  $\gamma_1 = 1/\sqrt{n}$ .

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<sup>1</sup>L. R. SALVOSA, "Tables of Pearson's type III function," *Ann. Math. Stat.*, v. 1, 1930, following p. 198.

21[K].—J. W. WHITFIELD, "The distribution of the difference in total rank value for two particular objects in  $m$  rankings of  $n$  objects," *British Jn. of Stat. Psychology*, v. 7, 1954, p. 45-49.

Let  $r_{ij}$  ( $i = 1, \dots, m$ ;  $j = 1, \dots, n$ ) be  $m$  rankings of  $n$  individuals, where the total ranks assigned to two particular individuals  $a$  and  $b$  are of interest.

The author considers the statistic  $d = \sum_{i=1}^m (r_{ia} - r_{ib})$ , the difference in total rank

values, and obtains the one-sided cumulative distribution function  $\frac{1}{2}P[|d| \geq k]$  on the assumption of randomness to 5D for  $n = 2, 3, n = 3(1)8$ ;  $m = 4, n = 3(1)7$ ;  $m = 5, n = 3(1)5$ ;  $m = 6, n = 3, 4$ ;  $m = 7, 8, n = 3$ . The first four moments of  $d$  and a normal approximation are also given.

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22[K].—WM. R. THOMPSON, *Tables of the Four Variable N- and Psi-Functions*. 9 + 88 typewritten pages (ozalided), deposited in the UMT FILE.

Let  $n = r + s$ ,  $n' = r' + s'$  and define

$$\begin{aligned} N(r, s, r', s') &= \binom{n + n' + 2}{n + 1} \psi(r, s, r', s') \\ &= \sum_{t=0}^{t \leq r, r'} \binom{r + r' + 1}{r' - t} \binom{s + s' + 1}{s - t}, \end{aligned}$$

where  $\binom{k}{m}$  is the binomial coefficient.

These tables give exact values of  $N$  for appropriate positive integral arguments such that  $3 \leq n \leq 20$ ,  $2 \leq n' \leq 20$ ,  $n \geq n'$  and 7S approximations (in some cases exact) of  $10^7\psi$  for appropriate non-negative integral arguments such that  $1 \leq n \leq 20$ ,  $0 \leq n' \leq 20$ ,  $n \geq n'$ .

The word appropriate here refers to the fact that a user of the table may take advantage of identities which result from permuting the arguments in certain

ways. A nine-page explanation of the tables and their method of construction is included.

The tables include, as subheadings in each "cell" of the table, parenthesized values of  $n$ ,  $n'$ , and  $D = \left( \frac{n + n' + 2}{n + 1} \right)$ . The author believes the tables to be error-free.

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23[L].—NBS Applied Mathematics Series, No. 32, *Table of Sine and Cosine Integrals for arguments from 10 to 100*. U. S. Government Printing Office, Washington, D. C., 1954, xvi + 187 p., 20.5 × 27 cm. Price \$2.25.

The first edition<sup>1</sup> of this volume appeared in 1942. In the present second edition the bibliography has been brought up to date, the table of  $p(1 - p)$  has been replaced by a table of  $p(1 - p)/2$ , the table of  $E_2$  and  $F_2$  has been added.

The principal table (180 p.) gives 10D values of

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \quad \text{Ci}(x) = \int_x^\infty \frac{\cos t}{t} dt$$

with second central differences for  $x = 10(.01)100$ . Interpolation by EVERETT's formula gives accuracy to within 1.2 units of the tenth decimal place.

Auxiliary tables:  $n\pi/2$  for  $n = 1(1)100$ ; 15D.  $p(1 - p)/2$  for  $p = 0(.001)1$ ; exact values. The EVERETT coefficients  $E_2(p)$  and  $F_2(p)$  for  $p = 0(.001)1$ ; 7D.

In the Introduction A. N. LOWAN gives the fundamental formulas for these functions, the method of computation and the preparation and checking of the manuscript; and describes both direct and inverse interpolation and their accuracy. A bibliography of tables, references to applications, and a list of texts and handbooks is also given.

There are graphs of  $\text{Si}(x)$  and  $\text{Ci}(x)$ , a Preface by A. V. ASTIN, and a Foreword by J. A. STRATTON.

A. E.

<sup>1</sup> NYMTP, "Table of sine and cosine integrals for arguments from 10 to 100." New York, 1942.

24[L].—S. RUSHTON, "On the confluent hypergeometric function  $M(\alpha, \gamma, x)$ ," *Sankhyā*, v. 13, 1954, p. 369-376.

S. RUSHTON & E. D. LANG, "Tables of the confluent hypergeometric function." *Sankhyā*, v. 13, 1954, p. 377-411.

The second of these papers gives 7S values of

$$M(\alpha, \gamma, x) = \sum_{j=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(\gamma + j)} \frac{x^j}{j!}$$

for  $\gamma = .5(.5)3.5, 4.5$ ;  $x = .02(.02).1(.1)1(1)10(10)50, 100, 200$ , and an extensive range of integer and half-integer values of  $\alpha$ . This range varies between  $0 \leq \alpha \leq 25$  and  $0 \leq \alpha \leq 50$ , and the interval is .5 or 1.

In the first paper the author describes the principal properties of the confluent hypergeometric function, its application to facilitate certain sequential tests of composite hypotheses, and the construction of the tables. When  $\alpha = \gamma$  or  $\gamma + 1$ , the confluent hypergeometric function can be expressed in terms of the exponential function, and from here the recurrence relations enable the computer to proceed to other values of  $\alpha$ , as long as  $\alpha - \gamma$  is an integer. In other cases, the power series expansion was used for  $x < 5$ , and the asymptotic expansion, improved by Airey's converging factor, for  $x \geq 5$ .

Companion tables<sup>1</sup> for  $\gamma = 3, 4$  were reviewed in RMT 1003 (*MTAC*, v. 6, 1952, p. 155-156).

A. E.

<sup>1</sup> P. NATH, "Confluent hypergeometric function," *Sankhyā*, v. 11, 1951, p. 153-166.

25[L].—MILTON ABRAMOWITZ & PHILIP RABINOWITZ, "Evaluation of Coulomb wave functions along the transition line," *Phys. Rev. (2)*, v. 96, 1954, p. 77-79.

Work on Coulomb wave functions at the NBS Computation Laboratory has been reviewed in RMT 1091 [*MTAC*, v. 7, p. 101-102], UMT 186 [*MTAC*, v. 8, p. 97], and RMT 1249 [*MTAC*, v. 8, p. 224]. In continuation of this work, the authors derive asymptotic expansions of  $F_0, F_0', G_0, G_0'$  for  $\rho = 2\eta$ , in descending powers of  $\beta = (\frac{2}{3}\eta)^{\frac{1}{2}}$ . They also indicate the computation of  $F_0$  and  $G_0$  for  $\rho$  near  $2\eta$  by means of Taylor series.

The paper contains 7D tables of  $F_0, F_0', G_0, G_0'$  for  $\rho = 2\eta = 0(.5)20(2)50$ . The tabular values were computed to 9D on the SEAC of the NBS by means of a program prepared by C. E. Fröberg and based on numerical evaluation of integral representations. The 7D given in the tables are stated to be correct to within one unit of the last place, and five-point Lagrangian interpolation will yield full accuracy for  $\rho \geq 3$ .

In a companion paper,<sup>1</sup> expansions of Coulomb wave functions (for any  $L$ ) are obtained for the region  $0 < \rho < 2\rho_1 = 2\eta + 2[\eta^2 + L(L+1)]^{\frac{1}{2}}$ .

A. E.

<sup>1</sup> MILTON ABRAMOWITZ & H. A. ANTOSIEWICZ, "Coulomb wave functions in the transition region," *Phys. Rev. (2)*, v. 96, 1954, p. 75-77.

26[L].—J. CLUNIE, "On Bose-Einstein functions," *Phys. Soc. Proc. Sect. A*, v. 67, 1954, p. 632-636.

The author provides formulas useful for the computation of the function

$$(1) \quad G_k(\eta) = \int_0^\infty \frac{x^k dx}{e^{x^2} - 1}$$

(the Cauchy Principal Value of the integral is to be taken when  $\eta > 0$ ): these formulas include asymptotic expansions for large  $|\eta|$  and power series expansions for small  $|\eta|$ , and the formula

$$(2) \quad G_k(\eta) = \sum_{r=0}^{n-1} 2^{-k-r} F_k(2^r \eta) + 2^{-k-n} G_k(2^n \eta)$$

where  $F_k$  is the Fermi-Dirac function.<sup>1</sup>

4D numerical tables of  $G_1(\eta)$  are given for  $\eta = -3(.2) - .6(.1).6(.2)20$ : outside this range the asymptotic formulas are valid. The values were computed from (2) and available tables<sup>1</sup> of  $F_1$ , they were checked by differencing and, in the interval  $-.5(.1).5$ , also by independent computation from the power series; and it is stated that the error in the present tables is less than 1 unit in the fourth decimal place.

The power series expansion of  $G_2$  and graphs of multiples of  $G_{-1}$ ,  $G_1$ ,  $G_1$  were given by ROBINSON.<sup>2</sup>

A. E.

<sup>1</sup> J. McDougall & E. C. Stoner, *Roy. Soc. Phil. Trans.*, v. 237A, 1938, p. 67-104.

<sup>2</sup> J. E. Robinson, *Phys. Rev.*, s. 2, v. 83, 1951, p. 678-679.

27[L].—JAMES G. BERRY, *Tables of Some Functions Related to the Legendre Functions  $P_n^{-m}(x)$  and  $Q_n(x)$  when  $n$  is a complex number*. Two copies, each 28 pages of copied typescript, deposited in the UMT FILE.

Tables of  $\left(\frac{1-x}{1+x}\right)^{-m/2} P_n^{-m}(x)$  and of  $\left(\frac{1-x}{1+x}\right)^{-m/2} \frac{d}{dx} [P_n^{-m}(x)]$ , for

$m = 0(1)20$ , and of  $Q_n(x)$  and  $\frac{d}{dx} [Q_n(x)]$ , where  $P_n^{-m}(x)$  is the associated Legendre function of the first kind and  $Q_n(x)$  is the Legendre function of the second kind, for  $n = -0.5 \pm 10.24595735 \pm i(10.18477501)$ . The range of  $x$  is .000(.032).960(.002).998, and  $x = .999875$ .

Real and imaginary parts of the two functions are in most cases given to 8S, although only spot checks were made and the author admits some instances in which the error is  $\pm 5$  in 7th S. Computation was done by a modification of the Runge-Kutta method on MIDAC, in connection with the author's Ph.D. Dissertation in Engineering Mechanics.

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28[L].—J. BERGHUIS, *A Table of Some Integrals*. R 245, *Mathematisch Centrum, Amsterdam*. Eight page mimeographed typescript, deposited in the UMT FILE.

This table contains

$$8D \text{ values of } f_n(x) = \int_0^x v^n \tan v \, dv, \quad n = 1(1)5, \quad x = 0(0.05)1.50$$

$$8D \text{ values of } g_n(x) = \int_0^x v^n \cot v \, dv, \quad n = 1(1)5, \quad x = 0(0.05)2.50$$

$$7D \text{ values of } F_n(x) = \int_0^x v^n \tanh v \, dv, \quad n = 1(1)4, \quad x = 0(0.02)1.98$$

$$7D \text{ values of } G_n(x) = \int_0^x v^n \coth v \, dv, \quad n = 1(1)4, \quad x = 0(0.02)1.98.$$

The error is stated to be  $10^{-8}$  in the last 3 functions,  $3 \times 10^{-8}$  in  $f_n(x)$ .  $F_n(x)$  and  $G_n(x)$  were computed by the ARRA; the other functions were hand computed and checked by differencing on a National Accounting machine class 31.

29[L].—Staff of the Computation Department of Mathematisch Centrum, Amsterdam, *Table of Polylogarithms, Report R24, Part I: Numerical Values*. 53 mimeographed pages deposited in the UMT FILE.

In three tables are given 10D values of  $F_n(z)$ , defined by  $\sum_{k=1}^{\infty} k^{-n} z^k$  for  $|z| < 1$  and for other values of  $z$  by analytic continuation, called polylogarithms of order  $n$  and argument  $z$ . In all 3 tables,  $n = 1(1)12$ .

Table I:  $z = x; x = -1(0.01)1$

Table II:  $z = ix; x = 0(0.01)1$

Table III:  $z = e^{i\alpha/2}; \alpha = 0(0.01)2$ .

The maximum error is stated to be  $10^{-10}$ .

30[S].—ANN T. NELMS, "Graphs of the Compton energy-angle relationship and the Klein-Nishina formula from 10 Kev to 500 Mev.," National Bureau of Standards Circular 542, 1953, iv + 89 p.

This Circular contains eight principal graphs, most of them with a number of subsidiary graphs on a larger scale to give increased accuracy.

Fig. I. Scattered photon energy versus angle,

$$h\nu = \frac{h\nu_0}{1 + \alpha_0(1 - \cos \theta)}, \quad \alpha_0 = \frac{h\nu_0}{mc^2}.$$

Each curve gives  $h\nu$  as a function of  $\theta$ , for a constant initial photon energy  $h\nu_0$ .

Fig. II. Recoil energy versus angle,

$$T = \frac{2\alpha_0 h\nu_0}{1 + 2\alpha_0 + (1 + \alpha_0)^2 \tan^2 \psi},$$

$T$  as a function of  $\psi$ , for constant  $h\nu_0$ .

Fig. III. Photon wave length distribution,

$$f(\lambda_0, \lambda) = \frac{3}{8} \left( \frac{\lambda_0}{\lambda} \right)^2 \left[ \frac{\lambda_0}{\lambda} + \frac{\lambda}{\lambda_0} - 2(\lambda - \lambda_0) + (\lambda - \lambda_0)^2 \right]$$

as function of  $\lambda$ , for constant  $\lambda_0$ .

Fig. IV. Photon angular distribution,  $(2\pi)^{-1} \sigma_0 f$  as function of  $\theta$ , where  $h\nu = mc^2/\lambda$  and  $\theta$  are connected as in Fig. I. Each curve is plotted for constant  $h\nu_0$ . The last subsidiary graph gives  $h\nu_0$  as a function of the angle at which  $\sigma_0 f$  is minimum.

Fig. V. Electron angular distribution,

$$\frac{\sigma_0 f(\lambda_0, \lambda)}{2\pi} \frac{(1 + \alpha_0)^2 (1 - \cos \theta)^2}{\cos^3 \psi}, \text{ where } \tan \psi = \frac{1}{1 + \alpha_0} \left( \frac{2\alpha_0 \alpha}{\alpha_0 - \alpha} - 1 \right)^{1/2},$$

as a function of  $\psi$ , for constant  $h\nu_0$ .

Fig. VI. Photon energy distribution,

$$\frac{\sigma_0 \lambda^2}{mc^2} f(\lambda_0, \lambda)$$

as a function of  $h\nu$ , for constant  $h\nu_0$ .

Fig. VII. Electron energy distribution, same quantity as in VI as a function of  $T$ , for constant  $h\nu_0$ .

Fig. VIIIa. Total Compton cross section and effective cross section as functions of  $h\nu_0$ , Fig. VIIIb. Fraction of incident energy absorbed as function of  $h\nu_0$ .

It is stated that all calculations and curves are accurate to 1 per cent, and that the subsidiary graphs are such that interpolated values can be obtained in general to 2 per cent accuracy.

The circular is one of a series of surveys and tabulations of information on radiation physics.

A. E.

# TABLE ERRATA

242.—R. S. BURINGTON, *Handbook of Mathematical Tables and Formulas*. 3rd Edition. Handbook Publishers, Inc., Sandusky, Ohio, 1953.

I have recently checked, by differentiation, all of the indefinite integrals in this edition of the *Handbook*. The following errors were discovered. They are also present in the 2nd edition.

P. 68, no. 146. In the next to the bottom line of the page,  
for  $(m + np + n)$ , read  $(m + np + n)a$ .

P. 71, no. 177. In the  $\tan^{-1}$  form, insert the restriction:  $a > 0, c < 0$ .  
In the  $\tanh^{-1}$  form, insert the restriction:  $a > 0, c > 0$ .

P. 71, no. 178. For  $+\frac{(ad - bc)^2}{8ac}$ , read  $-\frac{(ad - bc)^2}{8ac}$ .

P. 73, no. 195. Insert the restriction:  $b > 0$ .

P. 75, no. 225. Insert the restriction:  $b > 0$ .

P. 75, no. 226. The numerator,  $-1$ , of the coefficient of  $\sin^{-1} U$  should be replaced by:  $\text{Sgn}(d \cos ax - c \sin ax)$ , where  $\text{Sgn } z = 1$ , for  $z > 0$ ,  $-1$  for  $z < 0$ , and  $= 0$  for  $z = 0$ .

The expression for  $U$  should read:

$$U = \left[ \frac{c^2 + d^2 + b(c \cos ax + d \sin ax)}{\sqrt{c^2 + d^2} |b + c \cos ax + d \sin ax|} \right].$$

The final restriction,  $-\pi < ax < \pi$ , is unnecessary.

P. 78, no. 258. The restriction should read:  $b > 0, b > c, \cos ax > 0$ .

Most persons using integral tables are aware of errors, which are common to many tables of integrals, such as the following.

The first type is illustrated by the example:  $\int \frac{dx}{x} = \log x$ . A correct form for this integral would be  $\int \frac{dx}{x} = \log |x|$ . In this latter form, negative values of  $x$  may be used.

The second type is illustrated by the example:  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$ . This form is not valid for  $a < 0$ . A form which is valid for all  $a \neq 0$  may be obtained from this form by replacing the  $a$  by  $|a|$ .

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243.—E. JAHNKE & F. EMDE, *Tables of Functions*. Fourth Edition, 1945, New York and earlier editions.

On p. 262, for  $h_1(0.1) = 6.118$  read  $h_1(0.1) = 6.342$ .

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Remark: Errors in this volume have been noted earlier in *MTAC* as follows: v. 1, p. 198, 390; v. 2, p. 47, 350; v. 3, p. 41, 314, 364 (review), 423; v. 6, p. 196 (review 990[L]), 237.

240.—See item 2 of the Corrigenda.

## NOTES

### A Conference on Mathematical Tables

A conference on mathematical tables was held at the Massachusetts Institute of Technology on September 15 and 16, 1954, under the leadership of Professor P. M. MORSE. The following excerpts from his summary will be of interest to readers of *MTAC*. They were written by Professor MORSE.

This conference, under the joint auspices of the National Science Foundation and the Massachusetts Institute of Technology, was held to discuss the needs, in this country, for Tables of Mathematical Functions, in the light of recent developments in high-speed computers. Twenty-eight persons attended the two-day sessions and took part in round-table discussions on the general topics: Future Need for Tables; What Form Should Tables Have?; What Functions Need Tabulating?; and What Should Be Done About It?

### General Conclusions

**Still Need for Tables.** There was general agreement that the advent of high-speed computing equipment changed the task of table making but definitely did not remove the need for tables. There are still many scientists and engineers who have no more than desk computing machines to help in their calculations, and even where electronic computers are easily available many calculations, exploratory and otherwise, still have to be made "by hand." Here tables of usual type and format are still needed.

**Need for a Handbook for the Occasional Computer.** There was a general agreement that an outstanding need is for a "Computer's Handbook," with four- or five-place tables of usually encountered functions, together with a discussion of their analytic properties and a set of formulas and tables for interpolation and other techniques useful to the occasional computer. (This volume was characterized by several as an "enlarged and up-to-date JAHNKE-EMDE.") The National Bureau of Standards for several years has planned such a volume but has not been able to afford the five to ten man-years required to prepare, edit, and publish it. The present conference strongly recommended that the Bureau produce such a volume and suggests that it request the National Science Foundation for financial aid to achieve this end.

**Publication of Specialized Tables.** It was also agreed that computing machines now produce many tables of functions, incidental to the solution of specific problems, which would be of use to other workers but which remain in manuscript form or in a few copies. The National Research Council publication *Mathematical Tables and other Aids to Computation (M.T.A.C.)* publishes announcements of many such tables and keeps copies, which it will loan to interested persons. It was agreed by the conference that these tables would be much more useful if they were published, even if the publication were by photo-offset or similar process. It was recommended that *M.T.A.C.* extend its services by publishing yearly (or biannually) a selection of tables submitted to it. Accuracy of the tables would be the responsibility of the producer; the *M.T.A.C.* editors would exercise the sort of supervision over form and content which is done for the usual contribution to a scientific journal. The conferees are of the opinion that such a publication would eventually pay its way if the printing costs could be held down, but that initial cost should be defrayed by the National Science Foundation or by the N.R.C. and the Mathematical Societies Tables fund. It is suggested that the editorial board of *M.T.A.C.* (or the National Research Council Mathematical Tables Committee) be enlarged to provide assistance in this work; some of the additional members should represent tables users as well as table producers.

**"Tables" for Electronic Computers.** It was generally agreed that users of digital computing machines did not need mathematical tables of the usual form. However, those of the conferees familiar with the use of such equipment suggested that a collection of algorithms, from which subprograms could be devised for the computation of various known functions, when they are needed in a general machine program, would be of value. Such formulas or procedures could, perhaps, be collected and published by *M.T.A.C.* or by N.B.S. Computation Laboratory from time to time. There also was a present and continuing need for small, high-

accuracy tables of key values of functions used by machines for starting series expansions or interpolation formulas which could be incorporated into machine programs.

*Additional Suggestions.* There seemed to be some demand for a collection of tables in punched-card form which can be duplicated or tabulated on request. It was not clear, however, whether the present and proposed collection at Watson Laboratories and at U.C.L.A. are likely to be sufficient or not. Further study is suggested.

The need for a continuing Index of Tables was stressed. A new Index will shortly be published in England, an up-to-date revision of the *Index of Mathematical Tables*, by FLETCHER, MILLER, and ROSENHEAD. It was suggested that M.T.A.C. publish yearly supplements thereafter. The M.T.A.C. editorial board will take this suggestion under advisement.

The rest of the discussion took up the problems of machine production of tables, standards of accuracy, format, interpolation techniques, and the kinds of functions needing tabulation, reaching conclusions which will be of interest to table makers and to organizations supporting table publishing, but which cannot now be formulated as specific recommendations.

*Executive Committee.* Finally, it was proposed that a committee be formed to help implement the recommendations of this report, to cooperate with agencies that desire to carry the proposals forward, and to call other conferences if this appears desirable. This committee has the following composition:

- PHILIP M. MORSE, Chairman, Dept. of Physics, Mass. Inst. of Tech.
- M. ABRAMOWITZ, Div. 11.2, Nat. Bur. of Stand., Washington 25, D. C.
- J. H. CURTISS, 80 Waterman St., Providence 6, R. I.
- R. W. HAMMING, Bell Telephone Labs., Murray Hill, N. J.
- D. H. LEHMER, Dept. of Math., Univ. of Cal., Berkeley 4, Cal.
- C. B. TOMPKINS, Num. Anal. Research, Univ. of Cal., Los Angeles 24, Cal.
- J. W. TUKEY, Fine Hall, Box 703, Princeton, N. J.

Correspondence concerning this report should be addressed to one of these persons.

#### International Analogy Computation Meeting

The Belgian Society of Telecommunications and Electronic Engineers announced plans to hold an "International Analogy Computation Meeting" in Brussels between September 27 and October 1, 1955. Detailed information may be obtained from the Secretary of the Organizing Committee, P. GERMAIN, Institut de Physique appliquée, Université libre de Bruxelles, 50, avenue Franklin Roosevelt, BRUSSELS (Belgium).

#### The Royal Society Depository for Unpublished Mathematical Tables

The Mathematical Tables Committee of the Royal Society from time to time furnishes lists of tables accepted into its Depository of Unpublished Tables, and these lists along with short descriptions of the tables are published in the *Philo-*

*sophical Magazine* and the *Journal* of the London Mathematical Society. The list of tables accepted up to the end of 1953 may be found in the *Phil. Mag.*, s. 7, v. 45, 1937, p. 599-609, and in the *London Math. Soc.*, *Jn.*, v. 29, 1954, p. 504-512.

### Note on Arrangement of Material

The Editorial Committee is attempting to arrange the contents to make reference easier. Generally, the contents will consist of seven sections:

- Papers
- Technical Notes and Short Papers
- Reviews and Descriptions of Tables and Books
- Table Errata
- Notes
- Queries and replies
- Corrigenda.

In each section material will be grouped according to content approximately in order of the Classification of Tables used by *MTAC*.

The most radical change will be the grouping of reviews and notes concerning unpublished tables; this grouping is tried as a means of making reference more convenient, but it is somewhat influenced by the increasing difficulty of telling what has been "published" and what is "unpublished" but widely distributed.

Another change is the absorption of the portions of the journal devoted to automatic computing into the sections listed above.

No major change in editorial policy is contemplated and none should be deduced from the material in this issue.

C. B. T.

### CORRIGENDA

V. 8, p. 226, l. -13, for M. W. WILKES read M. V. WILKES.

$$l. -10, \text{ for } \text{Ch}(x, \chi) = x \sin \chi \int_0^x \exp(\chi - x \sin \chi / \sin \lambda) \operatorname{cosec}^2 \lambda d\lambda$$

$$\text{read } \text{Ch}(x, \chi) = x \sin \chi \int_0^x \exp(x - x \sin \chi / \sin \lambda) \operatorname{cosec}^2 \lambda d\lambda.$$

l. -4, for 70° read 90°.

$$l. -3, \text{ for } \text{Ch}(x, \chi) + \text{Ch}(x, \pi - \chi) = 2 \exp(x - x \sin \chi) \text{Ch}(x \sin \chi, \frac{1}{2}\pi)$$

$$\text{read } \text{Ch}(x, \chi) + \text{Ch}(x, \pi - \chi) = 2 \exp(x - x \sin \chi) \text{Ch}(x \sin \chi, \frac{1}{2}\pi).$$

V. 8, p. 227, l. -15, last column, for .7626582 read .7627582.



## CLASSIFICATION OF TABLES

- A. Arithmetical Tables. Mathematical Constants
- B. Powers
- C. Logarithms
- D. Circular Functions
- E. Hyperbolic and Exponential Functions
- F. Theory of Numbers
- G. Higher Algebra
- H. Numerical Solution of Equations
- I. Finite Differences. Interpolation
- J. Summation of Series
- K. Statistics
- L. Higher Mathematical Functions
- M. Integrals
- N. Interest and Investment
- O. Actuarial Science
- P. Engineering
- Q. Astronomy
- R. Geodesy
- S. Physics, Geophysics, Crystallography
- T. Chemistry
- U. Navigation
- V. Aerodynamics, Hydrodynamics, Ballistics
- Z. Calculating Machines and Mechanical Computation

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